Robust Hedging of Options on a Leveraged Exchange Traded Fund

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Aim: make statements about the price of options given very mild modelling assumptions

Incorporate market information by supposing the prices of vanilla call options are known

Typically want to know the largest/smallest price of an exotic option (Lookback option, Barrier option, Variance option, Asian option,...) given observed call prices, but with (essentially) no other assumptions on behaviour of underlying

This talk: options on Leveraged Exchange Traded Funds (LETF)

Why? Heavily traded, and interesting features to the solution!
Financial Setting

- Option priced on an asset \((S_t)_{t \in [0, T]}\), option payoff \(F((S_t)_{t \in [0, T]})\)
- Dynamics of \(S\) unspecified, but suppose paths are continuous, and we see prices of call options at all strikes \(K\) and at maturity time \(T\)
- Assume for simplicity that all prices are discounted — this won’t affect our main results
- Under risk-neutral measure, \(S\) should be a (local-)martingale, and we can recover the law of \(S_T\) at time \(T\), \(\mu\) say, from call prices \(C(K)\)
Leveraged Exchange Traded Fund (LETF)

- ETF attempts to match returns on a benchmark asset/index 1:1
- LETF attempts to match returns on a benchmark asset/index up to factor, e.g. 2:1 → 10% increase in index → 20% increase in LETF
- Over time, e.g. daily rebalancing leads to tracking errors
- Dynamics of the LETF with leverage ratio $\beta > 1$ are given by

$$L_t = S_t^\beta \exp \left( -\frac{\beta(\beta - 1)}{2} V_t \right),$$

$V_t$ is the accumulated quadratic variation of log $S_t$

- Eliminate $V_t$ by time change, $\tau_t := \inf\{s \geq 0 : V_s = t\}$ and $X_t := S_{\tau_t}$. So,

$$d\langle X \rangle_t = d\langle S \rangle_{\tau_t} = S_{\tau_t}^2 dV_{\tau_t} = X_t^2 dt$$

and $X_t$ is a geometric Brownian motion (GBM)
LETF model-independent pricing problem

Want to consider (maximum) price of call option on LETF under assumption that law of $S_T$ (under $\mathbb{Q}$) is known, but no other modelling assumption. Corresponds to:

**Main Problem**

Find

$$\sup_{\tau} \mathbb{E} \left[ \left( X^\beta_T \exp \left( -\frac{\beta(\beta - 1)}{2} \tau \right) - k \right)_+ \right], \quad \text{(LOptSEP)}$$

over stopping times $\tau$ such that $X_T \sim \mu$, where $X$ is a GBM

- Also: is there an arbitrage if the price of the option on the LETF exceeds this?
- This is a form of Optimal Skorokhod Embedding Problem (OptSEP)
Existing Literature

Rich literature on these problems:

- Starting with Hobson (’98) connection with Skorokhod Embedding problem → explicit optimal solutions for many different payoff functions (Brown, C., Dupire, Henry-Labordère, Hobson, Klimmek, Obłój, Rogers, Spoida, Touzi, Wang, . . .)

- Recently, model-independent duality has been proved by Dolinsky-Soner (’14):

\[
\sup_{\mathbb{Q}: S_T \sim \mu} \mathbb{E}^\mathbb{Q}[X_T] = \text{price of cheapest super-replication strategy}
\]

Here the super-replication strategy will use both calls and dynamic trading in underlying, and is model-independent. The sup is taken over measures \( \mathbb{Q} \) for which \( S \) is a martingale. (See also Hou-Obłój and Beiglböck-C.-Huesmann-Perkowski-Prömel)

- The problem of finding the maximising martingale \( S \) is commonly called the Martingale Optimal Transport problem (MOT)
Key observation of Beiglböck, C., Huesmann [BCH] (2016):

*Solutions to (OptSEP) are often characterised by simple geometric criteria*

Geometric criteria typically determined by the monotonicity principle ([BCH]):

*if I am better off ‘stopping’ a currently running path, and ‘transplanting’ the tail onto another stopped path (stopping at the same level), my solution is not optimal*

Monotonicity principle can be used to show that optimisers of (OptSEP) have a certain geometric form
Form of Optimiser: $K$-cave barrier

- Recall, problem is to maximise $\mathbb{E}[(M_\tau - k)_+]$, where $M_t = X_t^\beta e^{-\beta(\beta-1)t/2}$ is a martingale. Intuitively, aim to maximise local time of $M$ at $k$
- Can compute $M_t = k$ when $K(X_t) = t$, $K(x) = \frac{2}{\beta(\beta-1)} \ln(\frac{x^\beta}{k})$
- A $K$-cave barrier is a subset $\mathcal{R}$ of $\mathbb{R}_+ \times \mathbb{R}_+$ of the form $\mathcal{R} = \{(t, x) : t \leq \ell(x) \text{ or } t \geq r(x)\}$, where $\ell(x) \leq K(x) \leq r(x)$
- Similar concept (cave barrier) appeared in [BCH] ($K = \text{const}$)

**Theorem**

There exists an optimiser to $(LOptSEP)$ which is of the form

$$\tau_{\mathcal{R}} := \inf\{t > 0 : (t, X_t) \in \mathcal{R}\}$$

where $\mathcal{R}$ is a $K$-cave barrier.
$K$-cave barriers

$X_t$

$l(X_t)$  $r(X_t)$  $K(X_t)$

$t$  $\tau_R$
(Non-)uniqueness of Barriers

- Normally, at this point, a simple argument essentially due to Loynes would imply that there is a unique $K$-cave barrier with the right stopping distribution, which would then be the optimiser.

- However, for the $K$-cave barriers, there are generally multiple $K$-cave barriers which embed the same distribution; consider 3-atom measures. Crucial question:

  How to identify the optimal $K$-cave barrier?
PDE Heuristics for the Dual Solution

- We expect the Dual solution (superhedging portfolio) to take the form: \( \exists G, \lambda \) such that

\[
G(t, x) + \lambda(x) \geq F(t, x),
\]

where \( \lambda \) represents a portfolio of calls, \( F \) is the payoff of the option, and \( \gamma \) is the proceeds of a dynamic trading strategy in the underlying.

- We argue heuristically, inspired by arguments of Henry-Labordère: write \( F^\lambda(t, x) = F(t, x) - \lambda(x) \). Then we require:

\[
\mathcal{L}G := \frac{x^2}{2} \partial_x^2 G + \partial_t G \leq 0 \quad \text{and} \quad G \geq F^\lambda \quad \forall (t, x)
\]

and expect equality in PDE in \( \mathcal{R}^c \), and \( G = F^\lambda \) in \( \mathcal{R} \).

- Also conjecture smooth fit: \( \partial_t G = \partial_t F^\lambda = \partial_t F \) on boundaries

\[
\implies M := \partial_t G \text{ solves } \mathcal{L}M = 0 \text{ in } \mathcal{R}^c \text{ and } M = \partial_t F \text{ on } \partial \mathcal{R}
\]
In particular, we get: \( M(t, x) = \mathbb{E}^{(t,x)}[\partial_t F(X_{\tau_R}, \tau_R)] \), and integrating, we see that
\[
G(t, x) = \int_t^{r(x)} M(s, x) \, ds - Z(x)
\]
for some function \( Z \).

In fact, \( Z \) can be chosen (uniquely up to affine functions) in such a way to make \( G \) a martingale in \( \mathcal{R}^c \).

Now \( G(t, x) \geq F^\lambda(t, x) \) at \( t = \ell(x), t = r(x) \) implies that:

\[
\lambda(x) \geq Z(x) + \max\{0, F(\ell(x), x) - \int_{\ell(x)}^{r(x)} M(s, x) \, ds\} \\
\begin{aligned}
= & \Gamma(x) \\
& \text{for} \quad t = \ell(x) \quad \text{and} \quad t = r(x)
\end{aligned}
\]
Lemma (Easy)
Suppose $\mathcal{R}$ is a $K$-cave barrier which embeds $\mu$, and such that $\Gamma(x) = 0$ for all $x$. Then $\tau_{\mathcal{R}}$ is an optimiser of (LOptSEP).

Theorem (Hard)
There exists a $K$-cave barrier $\mathcal{R}$ which embeds $\mu$, and such that $\Gamma(x) = 0$ for all $x$. 

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$\Gamma$-condition: $\Gamma > 0$

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First step to proving the results:

**Theorem**

*The dual solution described above is indeed a dual solution (i.e. \( G \) is a martingale for some suitable \( Z \)).*

- Shown using essentially probabilistic techniques
- NB: No ‘explicit’ form for \( Z \)
- Clearly \( \Gamma = 0 \) is then a sufficient condition \( \iff \) primal = dual
- But: Not enough for theorem... Know (e.g. Dolinsky & Soner) that no duality gap, but don’t know optimal dual solution of the form above
Discretisation of Problem

- Idea: Take the original problem, and discretise time and space suitably (Random walk converging to the original Brownian motion)
- Can formulate the original problem in the discrete setting and formulate (LOptSEP) as a (countably infinite) linear programming problem
- Strong duality holds (in an appropriate sense) for the discretised problem, and can show existence of dual solutions, and natural condition corresponding to $\Gamma = 0$
- In the limit, exists optimal barrier, and embedding, and can make sense of $\Gamma = 0$ condition

$\implies$ Theorem holds
Conclusions

- Formulated the model-independent pricing problem for call on a Leveraged Exchange Traded Fund
- Corresponds to an interesting form of embedding problem: geometric characterisation does not guarantee uniqueness
- Need an additional condition, based on dual solution to determine optimal stopping region
- Proof of optimiser based on discretisation and