Cover’s universal portfolio, stochastic portfolio theory and the numéraire portfolio

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Motivation

In the article *Stochastic Portfolio Theory: an Overview* by R. Fernholz and I. Karatzas the following question was raised:

What, if any, is the connection between the theory of universal portfolios of T. Cover (1991) (discrete time) and F. Jamshidian (1992) (continuous time) and stochastic portfolio theory as initiated by R. Fernholz?
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What, if any, is the connection between the theory of universal portfolios of T. Cover (1991) (discrete time) and F. Jamshidian (1992) (continuous time) and stochastic portfolio theory as initiated by R. Fernholz?

This question stems from the fact that both theories ask for preference-free and general recipes how to choose a good (at least in the long run) long only portfolio among $d$ assets.
Cover’s universal portfolio - Overview

- Cover’s insight reveals the phenomenon that the “wisdom of hindsight” does not give any significant advantage as compared to a properly chosen “universal” portfolio which is constructed in a predictable way.

- The relevant optimality criterion here is the asymptotic growth rate of the portfolio

\[ \lim_{T \to \infty} \frac{1}{T} \log V_T, \]

where \((V_T)_{T \in T}\) denotes the wealth process and \(T\) stands either for \(\mathbb{N}\) (discrete time) or \([0, \infty)\) (continuous time).
Toy example

- Consider discrete time and denote by \( s = (s^1, \ldots, s^d)_{t=0}^\infty \) a trajectory of stock prices taking values in \( \mathbb{R}^d_{++} \).

- Fix \( T \in \mathbb{N} \) and think of an investor who at time \( T \) looks back and asks which stock she should have bought at time \( t = 0 \) by investing her initial endowment of 1 EUR and subsequently holding the stock.

- Obvious solution to this problem: pick the stock \( i \in \{1, \ldots, d\} \) which maximizes the performance \( \frac{s^i_T}{s^i_0} \).

- It clearly also maximizes the normalized logarithmic return

\[
\frac{1}{T} \left[ \log(s^i_T) - \log(s^i_0) \right] \quad i = 1, \ldots, d.
\]

- The “only” problem is, of course, that we have to make our choice at time \( t = 0 \) instead of \( t = T \).
**Toy example cont.**

- **Remedy:** simply divide at time $t = 0$ the initial endowment of 1 EUR into $d$ portions of $\frac{1}{d}$. At time $T$ your wealth equals

$$V_T = \frac{1}{d} \sum_{j=1}^{d} \frac{s_j^T}{s_j^0} \geq \frac{1}{d} \frac{s_i^T}{s_i^0},$$

where again $i$ denotes the stock which performed best during the time interval $[0, T]$.

- Passing again to normalized logarithmic returns we obtain

$$\frac{1}{T} \log(V_T) \geq \frac{1}{T} \left[ \log(s_i^T) - \log(s_i^0) - \log(d) \right].$$

- The difference between the retrospectively chosen portfolio and the “universal portfolio” consisting of equally weighing the $d$ stocks at time $t = 0$ can thus be estimated by $\frac{\log(d)}{T}$, which tends to zero as $T \to \infty$. 
Cover’s setting

- Model-free setting (no probability space) in discrete time $t \in \mathbb{N}$.
- Instead of only considering “pure” investments into one of the stocks as benchmark, Cover considers all constant rebalanced portfolio strategies:
- Let $b = (b^1, \ldots, b^d)$ be a fixed element of the $d$-dimensional closed simplex

$$\Delta^d = \left\{ x \in \mathbb{R}_+^d \mid \sum_{j=1}^d x^j = 1 \right\}.$$  

We denote by $\Delta^d$ the interior of the simplex.

- The corresponding portfolio wealth process $(V^b_t)_{t=0}^\infty$ is given by

$$\frac{V^b_{t+1}(s)}{V^b_t(s)} = \sum_{j=1}^d b^j \frac{s^j_{t+1}}{s^j_t}, \quad V^b_0 = 1$$

for each scenario $s = ((s^j_t)_{j=1}^d)_{t=0}^\infty$ of strictly positive numbers corresponding to stock prices.
Best retrospectively chosen portfolio

- For fixed $T$, define $V_T^*$ by
  \[ V_T^*(s) = \sup_{b \in \tilde{\Delta}^d} V_T^b(s), \]
  which is a function depending on the scenario $s = (s_t^1, \ldots, s_t^d)^T$.
  The optimizer is denoted by $b^*$ and is refereed to as best retrospectively chosen portfolio.

- Cover’s goal was to construct a “universal” portfolio chosen in a predictable way which performs as good as $(V_T^*)_{T=0}^\infty$ asymptotically for $T \to \infty$. 
Cover’s universal portfolio

- For a probability measure $\nu$ on $\Delta^d$ define portfolio weights as
  \[ b_T^\nu = \frac{\int_{\Delta^d} b V_T^b d\nu(b)}{\int_{\Delta^d} V_T^b d\nu(b)} \]

  which yield the following wealth process:
  \[ V_t(\nu)(s) = \int_{\Delta^d} V_t^b(s) d\nu(b). \]

  Thus, Cover’s universal portfolio consists in investing the portion $d\nu(b)$ of one’s wealth into the constant rebalanced portfolio with weights $b$.

- Note that the universal portfolio strategy at each time $T$ is built by averaging with a sort of posterior distribution of the form
  \[ \nu_T(A) = \frac{\int_A V_T^b d\nu(db)}{\int_{\Delta^d} V_T^b d\nu(b)} \]

  with prior distribution $\nu$ on $\Delta^d$ and the wealth at time $T$ interpreted as likelihood function.
Cover’s original result

**Theorem (Cover (91))**

Let \( \nu \) be a probability measure on \( \bar{\Delta}^d \) with full support. Then

\[
\lim_{T \to \infty} \frac{1}{T} \left( \log(V_T^*(s)) - \log(V_T(\nu)(s)) \right) = 0
\]

for all trajectories \( s = (s_1^t, \ldots, s_d^t)_{t=0}^{\infty} \) for which there are constants \( 0 < c \leq C < \infty \) such that

\[
c \leq \frac{s_{t+1}^j}{s_t^j} \leq C, \quad \text{for all} \quad j = 1, \ldots, d \quad \text{and all} \quad t \in \mathbb{N}.
\]
Extensions, Improvements and an (incomplete) literature overview

- **Quantitative estimates** when the distribution $\nu$ is specified.
- **Relaxation of the boundedness of the price relatives** (Cover and Ordentlich (96), Blum and Kalai (99)): For the uniform distribution on $\bar{\Delta}^d$, they obtain

$$\log(V_T(\nu)(s)) - \log(V_T^*(s)) \geq -(d - 1) \log(T - 1).$$

- Similar results in **continuous time** by Jamshidian (1992) for diffusions.
- Other **parametric families** instead of the constantly rebalanced one.
- Recent results by Wong (2015) on the **nonparametric family of long only functionally generated portfolios** of SPT in discrete time.
- Recently universal portfolio strategies have been studied extensively in an algorithmic and machine learning framework (Hazan and Kale (2015)).
The setting of stochastic portfolio theory

- **Stochastic portfolio theory (SPT)** is a theory for analyzing stock market structures and portfolio behavior and was introduced R. Fernholz.

- **Main quantity of interest:** *Relative performance with respect to the market portfolio.*

- **Economically speaking,** this amounts to take the *market portfolio* \( \sum_{j=1}^{d} s^j \) as *numéraire.*

- **One associates** to the stock prices \( (s^1, \ldots, s^d) \in \mathbb{R}^d_+ \) the vector of *market weights* \( (\mu^1, \ldots, \mu^d) \in \Delta^d \) by normalizing by the total market capitalization \( \sum_{j=1}^{d} s^j \) i.e.

\[
(\mu^1, \ldots, \mu^d) = \left( \frac{s^1}{\sum_{j=1}^{d} s^j}, \ldots, \frac{s^d}{\sum_{j=1}^{d} s^j} \right).
\]
Portfolio maps and relative wealth processes

- **A long only portfolio map** is a measurable function
  \[ \pi : \Delta^d \rightarrow \tilde{\Delta}^d \]
  which associates to the current market weights \( \mu_t = (\mu_1^t, \ldots, \mu_d^t) \) the weights \( (\pi(\mu_t) = (\pi_1^t(\mu_t), \ldots, \pi_d^t(\mu_t)) \) corresponding to the proportion of current wealth invested in the \( i \)th asset.

- **The constant rebalanced portfolio strategies** correspond to the constant functions \( \pi : \Delta^d \rightarrow \tilde{\Delta}^d \).

- **Relative wealth process**: \( Y^{\pi} = \frac{V^{\pi}}{V^{\mu}} \)
  
  ▶ **Discrete time (model-free)**: \( \frac{Y_{t+1}^{\pi}}{Y_{t}^{\pi}} = \sum_{j=1}^{d} \pi_j^t(\mu_t) \frac{\mu_{t+1}^j}{\mu_t^j} \)
  
  ▶ **Continuous time (at least in a semimartingale setting)**: \( \frac{dY_{t}^{\pi}}{Y_{t}^{\pi}} = \sum_{j=1}^{d} \pi_j^t(\mu_t) \frac{d\mu^j_t}{\mu_t^j} \)
Program of the remainder of this talk

- Consider instead of the constantly rebalanced portfolios larger classes of non-parametric portfolio maps $\pi : \Delta^d \to \bar{\Delta}^d$.

- Compare
  1. the best retrospectively chosen portfolio in this class of portfolio maps;
  2. the analog of Cover’s universal portfolio in this setting;
  3. the log-optimal portfolio within this class portfolio maps.

- Establish equal asymptotic growth rates for ergodic Markovian models for the market weights $\mu$ both in discrete and continuous time.

- “Modelfree” setup for comparing (1) and (2)
“Modelfree” setup for comparing (1) and (2) - Ingredients

Assumptions A

- \((\mathcal{G}, \| \cdot \|)\): compact set of functions generating the set of portfolio maps

\[
\mathcal{F} \mathcal{G} = \{ \pi : \Delta^d \to \bar{\Delta}^d, x \mapsto \pi(x) = \Pi(G)(x) \mid G \in \mathcal{G} \}
\]

via some function \(\Pi : \mathcal{G} \to \mathcal{F} \mathcal{G}\). The relative wealth corresponding to \(\Pi(G)\) is denoted by \(Y^{\Pi(G)} = Y^G\).

- For \(G \in \mathcal{G}\) and a given trajectory \((\mu_t)_{t \in \mathbb{T}}\) taking values in \(\Delta^d\), the wealth process \((Y^G_t)_{t \in \mathbb{T}}\) can be defined in a pathwise way, which is of course only an issue in continuous time.

- For every \(T\), \(G \mapsto Y^G_T\) is continuous.

- \(\nu\): Borel probability measure with full support on \(\mathcal{G}\)
“Modelfree” setup for comparing (1) and (2) - Types of portfolios

- **The best retrospectively chosen portfolio:** Define for each $T$ and a given trajectory $(\mu_t)_{t \in \mathbb{T}}$

  \[ Y_T^* = \sup_{G \in \mathcal{G}} Y_T^G. \]

  By compactness of $\mathcal{G}$ and continuity of $G \mapsto Y_T^G$ the optimizer exists and is denoted by $G_T^*$. 

- **The universal portfolio:** Define $\tilde{\nu} = \Pi_* \nu$ and

  \[ \pi_T^\nu = \frac{\int_{\mathcal{F}_G} \pi(\mu_T) Y_T^\pi d\tilde{\nu}(\pi)}{\int_{\mathcal{F}_G} Y_T^\pi d\tilde{\nu}(\pi)}, \]

  so that the relative wealth achieved by investing according to $\pi_T^\nu$ is given by

  \[ Y_T(\nu) = \int_{\mathcal{G}} Y_T^G d\nu(G). \]
Comparison between the best retrospectively and universal portfolio

The analog of Cover’s Theorem reads in the present setting as follows:

**Theorem (C., Schachermayer, Wong (2016))**

*Fix a Borel probability measure $\nu$ with full support on $\mathcal{G}$ and consider a trajectory $(\mu_t)_{t \in \mathbb{T}}$ taking values in $\Delta^d$. Suppose that Assumptions A holds and that for every $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$
\frac{1}{T} \log(Y_T^G) \geq \frac{1}{T} \log(Y_T^*) - \varepsilon
$$

for all $T \in \mathbb{T}$ and $G \in \mathcal{G}$ such that $\|G - G_T^*\| \leq \delta$. Then

$$
\lim_{T \to \infty} \frac{1}{T} \left( \log(Y_T^*) - \log(Y_T(\nu)) \right) = 0.
$$
*
Portfolio maps in discrete time

One possible choice for $G \equiv \mathcal{F}G$ in discrete time is the following set of functions:

- $\mathcal{L}^M$: set of all Lipschitz functions $\Delta^d \rightarrow \bar{\Delta}^d_{M-1}$ with Lipschitz constant $M > 0$, which pertains, e.g., to the metric defined by the norm $\| \cdot \|_1$ on $\bar{\Delta}^d$.

- Here $\bar{\Delta}^d_\epsilon$ denotes the set of $p \in \Delta^d$ verifying $p^j \geq \frac{\epsilon}{d}$, for $j = 1, \ldots, d$. 
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Corollary (C., Schachermayer, Wong (2016))

Fix a Borel probability measure $\nu$ with full support on $\mathcal{L}^M$. For every individual sequence $(\mu_t)_{t=0}^\infty$ in $\Delta^d$ we have

$$\lim_{T \to \infty} \frac{1}{T} (\log(Y^{*,M}_T) - \log(Y^M_T(\nu))) = 0.$$
Portfolio maps in continuous time - functionally generated portfolios

We consider the following set of concave functions for some fixed $M > 0$ and $0 < \alpha \leq 1$,

$$G^{M,\alpha} = \{ G \in C^{2,\alpha}(\Delta^d), \text{concave such that } \| G \|_{C^{2,\alpha}} \leq M$$

and $G \geq \frac{1}{M} \},$

where $C^{2,\alpha}(\Delta^d)$ denotes the Hölder space of 2-times differentiable functions from $\Delta^d \to \mathbb{R}$ whose derivatives are $\alpha$-Hölder continuous. That is,

$$C^{2,\alpha}(\Delta^d) = \{ G \in C^{2}(\Delta^d) | \| G \|_{C^{2,\alpha}} < \infty \},$$

where

$$\| G \|_{C^{2,\alpha}} = \max_{|k| \leq 2} \| D^k G \|_\infty + \max_{|k|=2} \sup_{x \neq y} \frac{|D^k G(x) - D^k G(y)|}{\| x - y \|_\alpha}$$

with $k$ denoting a multiindex.
Comparing (1) and (2)

Continuous time

Functionally generated portfolios as in SPT cont.

Lemma

Let $M, \alpha > 0$ be fixed. Then the set $G^{M,\alpha}$ is compact with respect to $\| \cdot \|_{C^{2,0}}$. 
Functionally generated portfolios as in SPT cont.

**Lemma**

Let $M, \alpha > 0$ be fixed. Then the set $G^{M,\alpha}$ is **compact** with respect to $\| \cdot \|_{C^{2,0}}$.

To the set of generating functions $G^{M,\alpha}$ we associate now the set of functionally generated portfolios $\mathcal{F}G^{M,\alpha}$ defined via

$$\mathcal{F}G^{M,\alpha} = \left\{ \pi : \Delta^d \rightarrow \bar{\Delta}^d, \right.$$ \[ x \mapsto \pi^i(x) = x^i \left( \frac{D^i G(x)}{G(x)} + 1 - \sum_{j=1}^{d} \frac{D^j G(x)}{G(x)} x^j \right), \quad | G \in G^{M,\alpha} \right\}. $$
The “modelfree” Master equation

Under the assumption

Assumption ((QV))

The path $\mu$ admits a continuous $S^+_d$-valued quadratic variation $[\mu]$ along $(\mathbb{T}_n)$ in the sense of Foellmer, i.e., for any $1 \leq i, j \leq d$ and all $t \geq 0$ the sequence

$$\sum_{s \in \mathbb{T}_n, s \leq t} (\mu_{s'}^i - \mu_s^i)(\mu_{s'}^j - \mu_s^j)$$

converges to a finite limit, denoted $[\mu^i, \mu^j]_t$, such that $t \mapsto [\mu^i, \mu^j]_t$ is continuous.

and by applying Foellmer’s functional Itô calculus we get the following pathwise version of Fernholz’s master equation, which also follows from Schied et al:

$$Y_T^G = \frac{G(\mu_T)}{G(\mu_0)} e^{\mathcal{G}([0, T])}, \quad 0 \leq T < \infty,$$

where $\mathcal{G}(dt) = -\frac{1}{2G(\mu_t)} \sum_{i,j} D_{ij} G(\mu_t) d[\mu^i, \mu^j]_t$. 
Comparison between best retrospectively and universal portfolio

The analog of Cover’s Theorem reads in the present setting as follows:

**Corollary (C., Schachermayer, Wong (2016))**

Let $M, \alpha > 0$ be fixed and let $(\mu_t)_{t \geq 0}$ be a continuous path satisfying Assumption (QV) such that for all $i \in \{1, \ldots, d\}$

$$\lim_{T \to \infty} \frac{1}{T} [\mu^i, \mu^i]_T < \infty.$$

Consider a probability measure $\nu$ on $G^{M,\alpha}$ with full support. Then

$$\lim_{T \to \infty} \frac{1}{T} (\log Y^*_T^{M,\alpha} - \log Y_T^{M,\alpha}(\nu)) = 0.$$
Comparison with the log-optimal portfolio

- By definition the log-optimal portfolio requires a stochastic model for $\mu$.

Assumption ((D) - Discrete time)

The process $\mu$ is a time homogeneous, ergodic Markov process on $\Delta^d$. 

Assumption ((C))

The process $\mu$ is a time-homogeneous ergodic Markovian Itô-diffusion on $\Delta^d$ of the form

$$\mu_t = \mu_0 + \int_0^t c(\mu_s) \lambda(\mu_s) \, ds + \int_0^t \sqrt{c(\mu_s)} \, dW_s, \quad \mu_0 \in \Delta^d,$$

where $W$ denotes a $d$-dimensional Brownian motion, $\lambda$ a measurable function from $\Delta^d \rightarrow \mathbb{R}^d$, $c$ a measurable function from $\Delta^d \rightarrow \mathbb{S}^d_+$, and the following integrability condition

$$\int_0^T \lambda^\top(\mu_t) c(\mu_t) \lambda(\mu_t) \, dt < \infty \quad \text{P-a.s.}$$
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$$
\int_0^T \lambda^\top(\mu_t)c(\mu_t)\lambda(\mu_t)dt < \infty \quad \mathbb{P}\text{-a.s.}
$$
The log optimal portfolio

- The log-optimal portfolio in the set \( \mathcal{L}^M (G^M, \alpha \text{ respectively}) \). Define the log-optimal portfolio among \( \mathcal{L}^M (G^M, \alpha \text{ respectively}) \) by

\[
\hat{Y}_T^M = \sup_{\pi \in \mathcal{L}^M} \mathbb{E} [\log(Y^\pi_T)], \quad \text{and} \quad \hat{Y}_T^{M, \alpha} = \sup_{G \in G^M, \alpha} \mathbb{E} [\log(Y^G_T)],
\]

respectively. Note that the optimizer does not depend on \( T \) due to the time homogenous Markov property of \( \mu \).

- The global long only log-optimal portfolio: The global log-optimal portfolio over all long only strategies is defined analogously and denoted via \( \hat{Y} \).
Equality of the asymptotic performance - discrete time

Theorem (C., Schachermayer, Wong (2016))

Let \( \mu = (\mu_t)_{t=0}^\infty \) be a \( \Delta^d \)-valued stochastic process satisfying Assumption D and the assumption of a finite expected yield of the log-optimal portfolio. Then we have the following equality \( \mathbb{P}\text{-a.s.} \):

\[
\liminf_{T \to \infty} \frac{1}{T} \log(Y^{*,M}_T) = \liminf_{T \to \infty} \frac{1}{T} \log(Y_M^T(\nu)) = \lim_{T \to \infty} \frac{1}{T} \log(\hat{Y}^M_T).
\]

In addition, the first equality holds true, for all sequences \( (\mu_t(\omega))_{t=0}^\infty \) in \( \Delta^d \).

Remark

Due to the assumption of ergodicity the above asymptotic growth rates are equal to a constant. This can be weakened as long as \( \lim_{T \to \infty} \frac{1}{T} \log(\hat{Y}^M_T) \) exists and some integrability conditions are satisfied.
Equality of the asymptotic performance - discrete time

To formulate a result not depending explicitly on the constant $M$, we define a universal portfolio $\hat{Y}(\nu) = (\hat{Y}_t(\nu))_{t=0}^{\infty}$ in the following way. For $M = 1, 2, 3, \ldots$ choose a measure $\nu^M$ on $\mathcal{L}^M$ with full support. Define $\nu = \sum_{M=1}^{\infty} 2^{-M} \nu^M$ and the process $Y(\nu)$ by

$$Y_t(\nu) = \int_{\bigcup_{M=1}^{\infty} \mathcal{L}^M} Y_t^\pi d\nu(\pi), \quad t \in \mathbb{N}.$$ 

**Corollary**

Under the assumptions of the above Theorem we have $\mathbb{P}$-a.s.

$$\lim_{M \to \infty} \lim_{T \to \infty} \frac{1}{T} \log(Y^*,M) = \lim_{T \to \infty} \frac{1}{T} \log(Y_T(\nu)) = \lim_{T \to \infty} \frac{1}{T} \log(\hat{Y}_T).$$
Equality of the asymptotic growth rates

Theorem (C., Schachermayer, Wong (2016))

Let $M, \alpha > 0$ be fixed and let $(\mu_t)_{t \geq 0}$ be a stochastic process satisfying Assumption (C). Moreover, suppose that

$$
\int_{\Delta_d} c_{ii}(x) \rho(dx) < \infty, \quad \text{for all } i \in \{1, \ldots, d\},
$$

$$
\int_{\Delta_d} \max_{i \in \{1, \ldots, d\}} |(c(x)\lambda(x))^i| \rho(dx) < \infty.
$$

Then

$$
\liminf_{T \to \infty} \frac{1}{T} \log Y^*_T, M, \alpha = \liminf_{T \to \infty} \frac{1}{T} \log Y^M_T, \alpha(\nu) = \lim_{T \to \infty} \frac{1}{T} \log \hat{Y}^M_T, \alpha, \quad \mathbb{P}\text{-a.s.}
$$
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\[
\int_{\Delta^d} c^{ii}(x) \rho(dx) < \infty, \quad \text{for all } i \in \{1, \ldots, d\},
\]

\[
\int_{\Delta^d} \max_{i \in \{1, \ldots, d\}} |(c(x)\lambda(x))|^i \rho(dx) < \infty.
\]

Then

\[
\liminf_{T \to \infty} \frac{1}{T} \log Y_{T}^{*, M, \alpha} = \liminf_{T \to \infty} \frac{1}{T} \log Y_{T}^{M, \alpha}(\nu) = \lim_{T \to \infty} \frac{1}{T} \log \hat{Y}_{T}^{M, \alpha}, \quad \mathbb{P}\text{-a.s.}
\]

A result independent of \( M \) yielding equality with the global log optimal portfolio can be achieved whenever it is functionally generated by some concave function \( G \in C^2(\Delta^d) \).
Conclusions

- Establish a link between stochastic portfolio theory and Cover’s universal portfolio by replacing Cover’s constantly rebalanced portfolios by more general portfolio maps $\pi : \Delta^d \mapsto \bar{\Delta}^d$.

- This yields equality of the asymptotic growth rates of
  1. the best retrospectively chosen portfolio;
  2. the analog of Cover’s universal portfolio in this setting;
  3. the log-optimal portfolio within this class portfolio maps (which is the global log-optimal under certain conditions);

for ergodic Markovian models for the market weights $\mu$, both in discrete and continuous time.
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Thank you for your attention!