At-the-money short-term asymptotics under stochastic volatility models

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Overview

The aim of study:

- To derive an asymptotic expansion of the implied volatility.
- To prove the validity of the expansion (estimate the error).
- Examine if a model is consistent to empirical facts.
- The framework should include rough volatility models.
- Make calibration more efficient.

The plan of talk:

- On asymptotic methods for the implied volatility.
- At-the-money, short-term asymptotic expansions.
- Asymptotic skew and curvature.
- The SABR and rough Bergomi models.
Stochastic volatility models

$(\Omega, \mathcal{F}, P)$: a probability space with a filtration $\{\mathcal{F}_t; t \in \mathbb{R}\}$. A log price process $Z$ is assumed to follow (under $Q$)

$$dZ_t = rd_t - \frac{1}{2}v_t dt + \sqrt{v_t} dB_t.$$ 

- $r \in \mathbb{R}$ stands for an interest rate,
- $v$ is a progressively measurable positive process with respect to a smaller filtration $\{\mathcal{G}_t; t \in \mathbb{R}\}$, $\mathcal{G}_t \subset \mathcal{F}_t$.
- The $\{\mathcal{F}_t\}$-Brownian motion $B$ is decomposed as

$$dB_t = \rho_t dW_t + \sqrt{1 - \rho_t^2} dW'_t,$$

where $W$ is an $\{\mathcal{G}_t\}$-BM and $W'$ is independent of $\{\mathcal{G}_t\}$.
- $\rho$ is a progressively measurable processes with respect to $\{\mathcal{G}_t\}$ and taking values in $(-1, 1)$.
A typical situation for stochastic volatility models is that \((W, W')\) is a two dimensional \(\mathcal{F}_t\)-Brownian motion and \(\mathcal{G}_t\) is the filtration generated by \(W\), that is,

\[
\mathcal{G}_t = \mathcal{N} \vee \sigma(W_s - W_r; r \leq s \leq t),
\]

where \(\mathcal{N}\) is the null sets of \(\mathcal{F}\).

Example: \(\rho \in (-1, 1)\) is constant and

\[
v_t = \exp(Y_t), \quad dY_t = \mu dt + \eta dW_t^H,
\]

where \(W^H\) is a fractional Brownian motion driven by \(W\) as

\[
W_t^H = c_H \int_{-\infty}^{t} (t - s)^{H-1/2} - (-s)^{H-1/2} dW_t.
\]

When \(H = 1/2\), then the model is the log-normal SABR.

Of course, \(\mathcal{G}_t\) can support a higher dimensional BM or Lévy.
The Black-Scholes implied volatility

An arbitrage-free price $p(K, \theta)$ of a put option at time 0 with strike $K > 0$ and maturity $\theta > 0$ is given by

$$p(K, \theta) = e^{-r\theta} E^Q[(K - \exp(Z_\theta))_+|\mathcal{F}_0]$$

$$= e^{-r\theta} \int_0^K Q(\log x \geq Z_\theta|\mathcal{F}_0)\,dx.$$

Denote by $p_{BS}(K, \theta, \sigma)$ the put option price with strike price $K$ and maturity $\theta$ under the Black-Scholes model with volatility parameter $\sigma > 0$.

Given a put option price $p(K, \theta)$, $K = Fe^k$, $F = e^{r\theta}$, the implied volatility $\sigma_{BS}(k, \theta)$ is defined through

$$p_{BS}(K, \theta, \sigma_{BS}(k, \theta)) = p(K, \theta).$$
Asymptotic analyses for stochastic volatility

1. Perturbation expansions:
   - Introduce an artificial parameter $\epsilon$ as $\nu_t = \nu_t^\epsilon$.
   - Consider
     \[
     \int_0^\theta \nu_t^\epsilon \, dt \to \int_0^\theta \nu_t^0 \, dt =: \sigma^2 \theta
     \]
     which is deterministic.
   - The limit model ($\epsilon \to 0$) is the Black-Scholes.
   - regular or singular: Yoshida’s martingale expansion works.
     a) small vol-of-vol (Lewis, Bergomi-Guyon)
     b) multi-scale (Fouque-Papanicolaou-Sircar-Solna)

2. Short-term (small time-to-maturity) asymptotics:
   - $\theta \to 0$.
   - The limit is degenerate:
     \[
     Z_\theta = r\theta - \frac{1}{2} \int_0^\theta \nu_t \, dt + \int_0^\theta \sqrt{\nu_t} \, dB_t = O(\theta) + O(\sqrt{\theta}).
     \]
   - After rescaling as $\theta^{-1/2} Z_\theta$, the limit model is the Bachelier.
Short-term asymptotics

1. Large deviation
   - Does not rescale $Z^\theta$.
   - For the transition density $p_\theta(x, y)$ of $Z^\theta$,
     \[-2\theta \log p_\theta(x, y) \to \inf \{ \|h\|_2^2; \varphi(0) = x, \varphi(1) = y, \dot{\varphi} = vh \}\]
   - Heat kernel expansion, the SABR formula, ...

2. Edgeworth expansion
   - Rescale $\theta^{-1/2}Z_\theta$ to get a normal limit law.
   - Yoshida, Kunitomo-Takahashi (small diffusion expansions)
   - Medvedev-Scaillet: rescale as $z = \frac{k}{\sigma_{BS}(k, \theta) \sqrt{\theta}}$ to expand $\sigma_{BS}(k, \theta)$.
   - Have a closer look around at-the-money $k = 0$.
   - Here, we give a rigorous approach under a mild condition (in particular, we do not rely on the Malliavin calculus).
The strategy

Stochastic expansion

\[ \downarrow \]

Characteristic function expansion

\[ \downarrow \]

Density expansion

\[ \downarrow \]

Put option price expansion

\[ \downarrow \]

Implied volatility expansion

\[ \downarrow \]

Asymptotic skew and curvature

Each parts are in fact not very new...

Watanabe, Yoshida, Kunitomo and Takahashi

Rem. a different approach in F. (2017), only for the 1st order.
The framework

Recall

\[ dZ_t = r dt - \frac{1}{2} \nu_t dt + \sqrt{\nu_t} dB_t, \quad dB_t = \rho_t dW_t + \sqrt{1 - \rho_t^2} dW'_t. \]

Denote by \( E_0 \) and \( \| \cdot \|_p \) respectively the expectation and the \( L^p \) norm under the regular conditional probability measure given \( \mathcal{F}_0 \), of which the existence is assumed.

Define the forward variance curve \( v_0(t) \) by

\[ v_0(t) = E_0[\nu_t] = E^Q[\nu_t|\mathcal{F}_0]. \]

We impose the following technical condition:

\[
\sup_{\theta \in (0,1)} \left\| \frac{1}{\theta} \int_0^\theta \nu_t dt \right\|_p < \infty, \quad \sup_{\theta \in (0,1)} \left\| \left\{ \frac{1}{\theta} \int_0^\theta \nu_t (1 - \rho_t^2) dt \right\}^{-1} \right\|_p < \infty.
\]
Stochastic expansion

Let

\[ M_\theta = \int_0^\theta \sqrt{v_t} dB_t, \quad \langle M \rangle_\theta = \int_0^\theta v_t dt, \quad \sigma_0(\theta) = \sqrt{\int_0^\theta v_0(t) du}. \]

We assume the following asymptotic structure: there exists a family of random vectors

\[ \{(M_\theta^{(0)}, M_\theta^{(1)}, M_\theta^{(2)}, M_\theta^{(3)}); \theta \in (0, 1)\}, \quad \sup_{\theta \in (0,1)} \|M_\theta^{(i)}\|_p < \infty \]

for all \( p > 0 \) and \( i = 1, 2, 3 \) such that \( \exists H > 0 \) and \( \exists \epsilon > 0 \),

\[
\lim_{\theta \to 0} \theta^{-2H-2\epsilon} \left\| \frac{M_\theta}{\sigma_0(\theta)} - M_\theta^{(0)} - \theta^H M_\theta^{(1)} - \theta^{2H} M_\theta^{(2)} \right\|_{1+\epsilon} = 0,
\]

\[
\lim_{\theta \to 0} \theta^{-H-2\epsilon} \left\| \frac{\langle M \rangle_\theta}{\sigma_0(\theta)^2} - 1 - \theta^H M_\theta^{(3)} \right\|_{1+\epsilon} = 0.
\]
Good to remember $E_0[M_\theta^2] = E_0[\langle M \rangle_\theta] = \sigma_0(\theta)^2 = O(\theta)$.

Further, we assume that the law of $M_\theta^{(0)}$ is standard normal for all $\theta > 0$ and the derivatives

$$a_\theta^{(i)}(x) = \frac{d}{dx} \left\{ E_0[M_\theta^{(i)}|M_\theta^{(0)} = x]\phi(x) \right\}, \quad i = 1, 2, 3,$$

$$b_\theta(x) = \frac{d^2}{dx^2} \left\{ E_0[|M_\theta^{(1)}|^2|M_\theta^{(0)} = x]\phi(x) \right\}$$

exist in the Schwartz space, where $\phi$ is the std. normal density.

Example: if $d \sqrt{v_t} = c(\sqrt{v_t})dW_t$, then by the Itô-Taylor,

$$M_\theta \approx \int_0^\theta (\sqrt{v_0} + c(\sqrt{v_0})W_t) + c'(\sqrt{v_0}) \int_0^t W_s dW_s)dB_t$$

$$= \sqrt{v_0} \theta^{1/2}$$

$$\times \left\{ \hat{B}_1 + \frac{c(\sqrt{v_0})}{\sqrt{v_0}} \theta^{1/2} \int_0^1 \hat{W}_t d\hat{B}_t + \frac{c'(\sqrt{v_0})}{\sqrt{v_0}} \theta \int_0^1 \int_0^t \hat{W}_s d\hat{W}_s d\hat{B}_t \right\}. $$
Characteristic function expansion I

Let $X_\theta = (Z_\theta - Z_0 - r\theta)/\sigma_0(\theta)$ and

$$Y_\theta = M^{(0)}_\theta + \theta^H M^{(1)}_\theta + \theta^{2H} M^{(2)}_\theta - \frac{\sigma_0(\theta)}{2} \left( 1 + \theta^H M^{(3)}_\theta \right).$$

**Lemma:** Under (2), for any $\alpha \in \mathbb{N} \cup \{0\}$,

$$\sup_{|u| \leq \theta^{-\epsilon}} |E_0[X_\theta^\alpha e^{iuX_\theta}] - E_0[Y_\theta^\alpha e^{iuY_\theta}]| = o(\theta^{2H+\epsilon}).$$

**Proof:** Since $|e^{ix} - 1| \leq |x|$, we have

$$|E_0[X_\theta^\alpha e^{iuX_\theta}] - E_0[Y_\theta^\alpha e^{iuY_\theta}]| \leq E_0[|X_\theta^\alpha - Y_\theta^\alpha|] + uE_0[|Y_\theta|^{\alpha} |X_\theta - Y_\theta|] \leq C(\alpha, \epsilon)(1 + |u|)\|X_\theta - Y_\theta\|_1 + \epsilon$$

for some constant $C(\alpha, \epsilon) > 0$. Since $\sigma_0(\theta) = O(\theta^{1/2})$, we obtain the result.  

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Characteristic function expansion II

**Lemma:** For any \( \delta \in [0, (H - \epsilon)/3) \) and any \( \alpha \in \mathbb{N} \cup \{0\} \),

\[
\sup_{|u| \leq \theta^{-\delta}} \left| E_0[ Y_{\theta}^\alpha e^{iuY_{\theta}}] \right|
\]

\[
- E_0 \left[ e^{iuM_{\theta}^{(0)}} \left( (M_{\theta}^{(0)})^\alpha + A(\alpha, u, M_{\theta}^{(0)}) + B(\alpha, u, M_{\theta}^{(0)}) \right) \right] = o(\theta^{2H+\epsilon}),
\]

where

\[
A_{\theta}(\alpha, u, x) = \left( iux^{\alpha} + \alpha x^{\alpha - 1} \right) (E_0[ Y_{\theta} | M_{\theta}^{(0)} = x] - x),
\]

\[
B_{\theta}(\alpha, u, x) = \left( -\frac{u^2}{2} x^{\alpha} + iux^{\alpha - 1} + \frac{\alpha(\alpha - 1)}{2} x^{\alpha - 2} \right) E_0[ |M_{\theta}^{(1)}|^2 | M_{\theta}^{(0)} = x].
\]

**Proof:** This follows from the fact that

\[
\left| e^{ix} - 1 - ix + \frac{x^2}{2} \right| \leq \frac{|x|^3}{6}
\]

for all \( x \in \mathbb{R} \).
Characteristic function expansion III

Lemma: Define $q_\theta(x)$ by

$$q_\theta(x) = \phi(x) - \theta^H a^{(1)}_\theta(x) - \theta^{2H} a^{(2)}_\theta(x)$$

$$- \frac{\sigma_0(\theta)}{2} (x\phi(x) - \theta^H a^{(3)}_\theta(x)) + \frac{\theta^{2H}}{2} b_\theta(x),$$

where $a^{(i)}_\theta$ and $b_\theta$ are defined by (3). Then,

$$\int_\mathbb{R} e^{iux} x^\alpha q_\theta(x) dx$$

$$= E_0 \left[ e^{iUM_\theta(0)} \left( (M_\theta(0))^\alpha + A(\alpha, u, M_\theta(0)) + B(\alpha, u, M_\theta(0)) \right) \right].$$

Proof: This follows from integration by parts. ////
Density expansion I

Lemma: For any $\alpha, j \in \mathbb{N} \cup \{0\}$,

$$\sup_{\theta \in (0,1)} \int |u|^j E_0[X_\theta^{\alpha} e^{iuX_\theta}] \, du < \infty$$

Proof: Since the distribution of $X_\theta$ is Gaussian conditionally on $\mathcal{G}_\theta$, it admits a density $p_\theta(x)$ under $Q(\cdot | F_0)$. Furthermore, the density function is in the Schwartz space $S$ by (1). Therefore,

$$\int |u|^j E_0[X_\theta^{\alpha} e^{iuX_\theta}] \, du = \int \left( \int u^j x^\alpha e^{iux} p_\theta(x) \, dx \right) \, du$$

$$= \int \left( \int e^{iux} \partial_x^j (x^\alpha p_\theta(x)) \, dx \right) \, du < \infty$$

since the Fourier transform is a map from $S$ to $S$. 

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Density expansion II

**Theorem:** The law of $X_\theta$ admits a density $p_\theta$ and for any $\alpha \in \mathbb{N} \cup \{0\}$,

$$\sup_{x \in \mathbb{R}} (1 + x^2)^\alpha |p_\theta(x) - q_\theta(x)| = o(\theta^{2H}) \quad (5)$$

as $\theta \to 0$, where $q_\theta$ is defined by (4).

**Proof:** As seen in the proof of Lemma, the density $p_\theta$ exists in the Schwartz space. By the Fourier identity

$$(1 + x^2)^\alpha |p_\theta(x) - q_\theta(x)|$$

$$= \frac{1}{2\pi} \left| \int \int e^{iux} (1 + y^2)^\alpha (p_\theta(y) - q_\theta(y)) dy e^{-iux} du \right|$$

$$= \frac{1}{2\pi} \left\{ \int_{|u| \leq \theta^{-\delta}} |\cdot| du + \int_{|u| \geq \theta^{-\delta}} |\cdot| du \right\}.$$
Combining the lemmas in the previous section, taking $\delta \in (0, \min\{\epsilon, (H - \epsilon)/3\})$, we have

\[
\int_{|u| \leq \theta^{-\delta}} \left| \int e^{iuy}(1 + y^2)^{\alpha}(p_\theta(y) - q_\theta(y))dy \right| du = o(\theta^{2H}).
\]

On the other hand,

\[
\int_{|u| \geq \theta^{-\delta}} \left| \int e^{iuy}(1 + y^2)^{\alpha} p_\theta(y)dy \right| du \leq \theta^{j\delta} \int_{|u| \geq \theta^{-\delta}} |u|^j E_0[(1 + X_\theta^2)^{\alpha} e^{iuX_\theta}]|du = O(\theta^{j\delta})
\]

for any $j \in \mathbb{N}$ by Lemma. The remainder

\[
\int_{|u| \geq \theta^{-\delta}} \left| \int e^{iuy}(1 + y^2)^{\alpha} q_\theta(y)dy \right| du
\]

is handled in the same manner.
Put option price expansion I

Denoting by \( p_\theta \) the density of \( X_\theta \) as before,

\[
\frac{p(Fe^{\sigma_0(\theta)z}, \theta)}{F^{\sigma_0(\theta)}} = e^{-r\theta} \int_{-\infty}^{z} \int_{-\infty}^{\zeta} p_\theta(x) dx e^{\sigma_0(\theta)\zeta} d\zeta.
\]

**Lemma:** Let \( q_\theta(x) \), \( \theta > 0 \) be a family of functions on \( \mathbb{R} \). If

\[
\sup_{x \in \mathbb{R}} (1 + x^2)^\alpha |p_\theta(x) - q_\theta(x)| = o(\theta^\beta)
\]

for some \( \alpha > 5/4 \) and \( \beta > 0 \), then for any \( z_0 \in \mathbb{R} \),

\[
\frac{p(Fe^{\sigma_0(\theta)z}, \theta)}{F^{\sigma_0(\theta)}} = e^{-r\theta} \int_{-\infty}^{z} \int_{-\infty}^{\zeta} q_\theta(x) dx e^{\sigma_0(\theta)\zeta} d\zeta + o(\theta^\beta)
\]

uniformly in \( z \leq z_0 \).
Put option price expansion II

Proof: By the Cauchy-Schwarz inequality,

\[ e^{-r\theta} \int_{-\infty}^{Z} \int_{-\infty}^{\zeta} |p_\theta(x) - q_\theta(x)| dze^{\sigma_0(\theta)\zeta} d\zeta \]

\[ \leq e^{-r\theta} \int_{-\infty}^{Z} \sqrt{\int_{-\infty}^{\zeta} \frac{dx}{(1 + x^2)^{2\alpha - 1}}} \]

\[ \times \sqrt{\int_{-\infty}^{\zeta} (1 + x^2)^{2\alpha - 1} |p_\theta(x) - q_\theta(x)|^2 dze^{\sigma_0(\theta)\zeta} d\zeta} \]

\[ \leq \pi e^{-r\theta + \sigma_0(\theta)Z} \sup_{x \in \mathbb{R}} (1 + x^2)^{\alpha} |p_\theta(x) - q_\theta(x)| \]

\[ \times \int_{-\infty}^{Z} \sqrt{\int_{-\infty}^{\zeta} \frac{dx}{(1 + x^2)^{2\alpha - 1}}} d\zeta, \]

which is \( o(\theta^\beta) \) if \( \alpha > 5/4 \).
Put option price expansion III

**Theorem:** Suppose we have (5) with $q_\theta$ of the form

$$q_\theta(x) = \phi(x) \left\{ 1 - \frac{\sigma_0(\theta)}{2} H_1(x) + \kappa_3(\theta)(H_3(x) - \sigma_0(\theta)H_2(x))\theta^H \right. + \left. \left( \kappa_4 H_4(x) + \frac{\kappa_3(\theta)^2}{2} H_6(x) \right)\theta^{2H} \right\} ,$$

where $H_k$ is the $k$th Hermite polynomial:

- $H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$, $H_4(x) = x^4 - 6x^2 + 3$, …

Then, for any $z_0 \in \mathbb{R}$,

$$p(Fe^{\sigma_0(\theta)z}, \theta) = \Phi(z) e^{\sigma_0(\theta)z} - \Phi(z - \sigma_0(\theta))$$

$$+ \phi(z) \left\{ \kappa_3 H_1(z) e^{\sigma_0(\theta)z} \theta^H + \left( \kappa_4 H_2(z) + \frac{\kappa_3^2}{2} H_4(z) \right)\theta^{2H} \right\} + o(\theta^{2H})$$

uniformly in $z \leq z_0$, where $\kappa_3 = \kappa_3(\theta)$. 
Implied volatility expansion

**Theorem:** Under the same condition as before,

\[
\sigma_{BS}(\sqrt{\theta}z, \theta) = \kappa_2 \left\{ 1 + \frac{\kappa_3}{\kappa_2} z \theta^H + \left( \frac{3\kappa_3^2}{2} - \kappa_4 + \frac{\kappa_4 - 3\kappa_3^2}{\kappa_2^2} z^2 \right) \theta^{2H} \right\} + o(\theta^{2H})
\]

when \( H < 1/2 \) and

\[
= \kappa_2 \left\{ 1 + \frac{\kappa_3}{\kappa_2} z \sqrt{\theta} + \left( \frac{3\kappa_3^2}{2} - \kappa_4 + \left( \frac{\kappa_4 - 3\kappa_3^2}{\kappa_2^2} + \frac{\kappa_3}{2\kappa_2} \right) z^2 \right) \theta \right\} + o(\theta)
\]

when \( H = 1/2 \), where \( \kappa_2 = \kappa_2(\theta) = \sigma_0(\theta)/\sqrt{\theta} \) and \( \kappa_3 = \kappa_3(\theta) \).

Note that \( \kappa_2(\theta) = \sqrt{\text{the averaged forward variance}} \).
Asymptotics for at-the-money skew and curvature

**Theorem:** Under the same condition as before,

\[ \partial_k \sigma_{BS}(0, \theta) = \kappa_3(\theta) \theta^{H-1/2} + o(\theta^{2H-1/2}), \]

\[ \partial_k^2 \sigma_{BS}(0, \theta) = \frac{2}{\kappa_2(\theta)} \frac{\kappa_4 - 3\kappa_3(\theta)^2}{\kappa_3(\theta)} \theta^{2H-1} + \kappa_3(\theta) \theta^{H-1/2} + o(\theta^{2H-1}). \]

**Proof:** Combine the previous expansions and

\[ \partial_k \sigma_{BS}(k, \theta) = \frac{Q(k \geq \sigma_0(\theta)X_\theta | \mathcal{F}_0) - \Phi(f_+(k, \theta))}{\sqrt{\theta} \phi(f_+(k, \theta))}, \]

\[ \partial_k^2 \sigma_{BS}(k, \theta) = \frac{p_\theta(k/\sigma_0(\theta))}{\sigma_0(\theta) \sqrt{\theta} \phi(f_+(k, \theta))} - \sigma_{BS}(k, \theta) \partial_k f_-(k, \theta) \partial_k f_+(k, \theta), \]

where

\[ f_\pm(k, \theta) = \frac{k}{\sqrt{\theta} \sigma_{BS}(k, \theta)} \pm \frac{\sqrt{\theta} \sigma_{BS}(k, \theta)}{2}. \]
The rough Bergomi model

Let $\rho_t = \rho \in (-1, 1)$ be a constant and

$$
\text{d} \log v_t = \eta \text{d} W_t^H + \text{deterministic drift},
$$

where $\eta > 0$ is a constant and $W^H$ is a fractional Brownian motion with the Hurst parameter $H \in (0, 1/2)$, given as

$$
W_t^H = c_H \int_{-\infty}^{t} (t - s)^{H-1/2} - (-s)^{H-1/2} \text{d} W_s
$$

with a normalizing constant $c_H > 0$. Since $v_t$ is log-normally distributed, (1) holds by Jensen’s inequality. We have

$$
v_t = v_0(t) \exp \left\{ \eta_H \sqrt{2H} \int_{0}^{t} (t - s)^{H-1/2} \text{d} W_s - \frac{\eta_H^2}{2} t^{2H} \right\},
$$

where $\eta_H = \eta c_H / \sqrt{2H}$. Note that $v_0(t)$ is rough.
The Hermite polynomials

Let $H_k, k = 0, 1, \ldots$ be the Hermite polynomials:

$$H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}$$

and $H_k(x, a) = a^{k/2} H_k(x/\sqrt{a})$ for $a > 0$. As is well-known, we have

$$\exp \left\{ ux - \frac{au^2}{2} \right\} = \sum_{k=0}^{\infty} H_k(x, a) \frac{u^k}{k!}$$

and for any continuous local martingale $M$ and $n \in \mathbb{N}$,

$$dL_t^{(n)} = nL_t^{(n-1)} dM_t, \quad (6)$$

where $L^{(k)} = H_k(M, \langle M \rangle)$ for $k \in \mathbb{N}$. 
Time-change for the rBergomi model

Define \( \hat{W}, \hat{W}', \hat{B} \) by

\[
\hat{W}_t = \frac{1}{\sigma_0(\theta)} \int_0^{\tau^{-1}(t)} \sqrt{v_0(s)} \, dW_s, \quad \hat{W}'_t = \frac{1}{\sigma_0(\theta)} \int_0^{\tau^{-1}(t)} \sqrt{v_0(s)} \, dW'_s
\]

and \( \hat{B} = \rho \hat{W} + \sqrt{1 - \rho^2} \hat{W}' \), where

\[
\tau(s) = \frac{1}{\sigma_0(\theta)^2} \int_0^s v_0(t) \, dt.
\]

Then, \((\hat{W}, \hat{W}')\) is a 2-dimensional Brownian motion under \( E_0 \) and for any square-integrable function \( f \),

\[
\int_0^a f(s) \, dW_s = \sigma_0(\theta) \int_0^{\tau(a)} \frac{f(\tau^{-1}(t))}{\sqrt{v_0(\tau^{-1}(t))}} \, d\hat{W}_t.
\]
Therefore,

\[ M_\theta = \sigma_0(\theta) \int_0^1 \exp \left\{ \theta^H F_t^t - \frac{\eta_H^2}{4} |\tau^{-1}(t)|^{2H} \right\} d\hat{B}_t \]

where

\[ F_u^t = \eta_H \sqrt{\frac{H}{2} \sigma_0(\theta) \theta^H} \int_0^u \frac{(\tau^{-1}(t) - \tau^{-1}(s))^{H-1/2}}{\sqrt{v_0(\tau^{-1}(s))}} d\hat{W}_s, \quad u \in [0, t]. \]

Let

\[ G_t^{(k)} = H_k(F_t^t, \langle F^t \rangle_t). \]

Then, we have

\[ M_\theta = \sigma_0(\theta) \int_0^1 \exp \left\{ \frac{-\eta_H^2}{8} |\tau^{-1}(t)|^{2H} \right\} \exp \left\{ \theta^H F_t^t - \frac{\theta^{2H}}{2} \langle F^t \rangle_t \right\} d\hat{B}_t \]

\[ = \sigma_0(\theta) \int_0^1 \exp \left\{ \frac{-\eta_H^2}{8} |\tau^{-1}(t)|^{2H} \right\} \sum_{k=0}^{\infty} G_t^{(k)} \frac{\theta^{Hk}}{k!} d\hat{B}_t. \]
Lemma: We have (2) with

\[ M^{(0)}_\theta = \hat{B}_1, \]
\[ M^{(1)}_\theta = \int_0^1 h_\theta(t) G_t^{(1)} \, d\hat{B}_t, \]
\[ M^{(2)}_\theta = \int_0^1 \left\{ \frac{\theta^2 H}{\theta^2 H} + h_\theta(t) \frac{G_t^{(2)}}{2} \right\} \, d\hat{B}_t, \]
\[ M^{(3)}_\theta = 2 \int_0^1 F_t^t \, dt, \]

where

\[ h_\theta(t) = \exp \left\{ -\frac{\eta_H^2}{8} |\tau^{-1}(t)|^{2H} \right\}. \]
Density expansion for the rBergomi model

Theorem: We have (5) with

\[ q_{\theta}(x) = \phi(x) \left\{ 1 - \frac{\sigma_{0}(\theta)}{2} H_{1}(x) + \kappa_{3}(\theta)(H_{3}(x) - \sigma_{0}(\theta)H_{2}(x))\theta^{H} \right. \]

\[ + \left. \left( \kappa_{4}H_{4}(x) + \frac{\kappa_{3}(\theta)^{2}}{2} H_{6}(x) \right) \theta^{2H} \right\}, \]

where

\[ \kappa_{3}(\theta) = \rho \eta_{H} \sqrt{\frac{H}{2}} \frac{1}{\theta^{H}\sigma_{0}(\theta)^{3}} \int_{0}^{\theta} \int_{0}^{t} (t - s)^{H-1/2} \sqrt{v_{0}(s)} ds v_{0}(t) dt, \]

\[ \kappa_{4} = \frac{(1 + 2\rho^{2})\eta_{H}^{2}H}{(2H + 1)^{2}(2H + 2)} + \frac{\rho^{2}\eta_{H}^{2}H\beta(H + 3/2, H + 3/2)}{2(H + 1/2)^{2}}. \]
Brownian bridge

Since $M^{(0)} = \hat{B}_1$, computing $E_0[M^{(i)}|M^{(0)} = x]$ reduces to compute expectations of iterated integrals of Brownian bridge.

Lemma:

$$
\hat{E} \left[ \int_0^1 \int_0^t f(s, t) \, d\hat{B}_s \, dt \right] = H_1(x) \int_0^1 \int_0^t f(s, t) \, ds \, dt,
$$

$$
\hat{E} \left[ \int_0^1 \int_0^t f(s, t) \, d\hat{B}_s \, d\hat{B}_t \right] = H_2(x) \int_0^1 \int_0^t f(s, t) \, ds \, dt,
$$

$$
\hat{E} \left[ \int_0^1 \left( \int_0^t f(s, t) \, d\hat{B}_s \right)^2 \, d\hat{B}_t \right] = H_3(x) \int_0^1 \left( \int_0^t f(s, t) \, ds \right)^2 \, dt
$$

$$
+ H_1(x) \int_0^1 \int_0^t f(s, t)^2 \, ds \, dt.
$$

and...
\[
\hat{E} \left[ \int_0^1 \left( \int_s^1 f(s,t) d\hat{B}_t \right)^2 ds \right] \\
= H_2(x) \int_0^1 \left( \int_s^1 f(s,t) dt \right)^2 ds + \int_0^1 \int_s^1 f(s,t)^2 dt ds, \\
\hat{E} \left[ \left( \int_0^t \int_0^s f(s,t) d\hat{B}_s d\hat{B}_t \right)^2 \right] = H_4(x) \left( \int_0^t \int_0^t f(s,t) ds dt \right)^2 \\
+ H_2(x) \int_0^1 \left( \int_0^t f(s,t) ds + \int_t^1 f(t,u) du \right)^2 dt \\
+ \int_0^1 \int_0^t f(s,t)^2 ds dt.
\]
Concluding remarks

- There is no technical difficulty to go higher orders.
- The same approach works for the small vol-of-vol perturbation.
- The rBergomi model explains the power law of volatility skew (and curvature).
- When the forward variance curve is flat, an expansion of the Forde-Zhang rate function of large deviation gives the same expansion of the implied volatility. Cf. Bayer et al.
- When the forward variance curve is flat, the (formal) small vol-of-vol (Bergomi-Guyon) expansion given by Bayer et al. (2016) coincides with our expansion.