Existence for a calibrated Regime Switching Local Volatility model

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Advances in Financial Mathematics, 11 January 2017
Outline

1. Processes matching given marginals
   - Motivation
   - Simulation of calibrated LSV models and theoretical results

2. A new fake Brownian motion
   - The studied problem
   - Main Result
   - Ideas of proof

3. Existence of Calibrated RSLV models
   - The calibrated RSLV model
   - Main results
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Motivation

Trying to match marginals

- The market gives the prices of European Calls $C(T, K)$ for $T, K > 0$ (idealized situation; in practice only \((C(T_i, K_i))_{1 \leq i \leq I}\)).
- A model \((S_t)_{t \geq 0}\) is calibrated to European options if

$$\forall T, K \geq 0, \quad C(T, K) = \mathbb{E} \left[ e^{-rT} (S_T - K)^+ \right].$$

- By Breeden and Litzenberger (1978), \{prices of European Call options for all $T, K > 0$\} $\iff \{\text{marginal distributions of } (S_t)_{t \geq 0}\}.$
- Dupire Local Volatility model (1992), matching market marginals:

$$dS_t = rS_t dt + \sigma_{Dup}(t, S_t) S_t dW_t$$

$$\sigma_{Dup}(T, K) = \sqrt{2 \frac{\partial_T C(T, K) + rK \partial_K C(T, K)}{K^2 \partial_{KK}^2 C(T, K)}}$$
Motivation: get processes with richer dynamics (e.g. take into account volatility risk) and satisfying marginal constraints.

Lipton (2002) and Piterbarg (2007),...: Local and Stochastic Volatility (LSV) model

\[ dS_t = rS_t + f(Y_t) \sigma(t, S_t) S_t dW_t \]

“Adding uncertainty” to LV models by a random multiplicative factor \( f(Y_t) \) where \( (Y_t)_{t\geq 0} \) is a stochastic process and \( f > 0 \).
Calibration of LSV Models

- By Gyongy’s theorem (1988), the LSV model is calibrated to $C(T, K), \forall T, K > 0$ if
  \[
  \mathbb{E} \left[ (f(Y_t)\sigma(t, S_t) S_t)^2 | S_t \right] = (\sigma_{Dup}(t, S_t) S_t)^2
  \]

\[
\sigma(t, x) = \frac{\sigma_{Dup}(t, x)}{\sqrt{\mathbb{E}[f^2(Y_t)|S_t = x]}}
\]

- The obtained SDE is nonlinear in the sense of McKean:
  \[
dS_t = rS_t dt + \frac{f(Y_t)}{\sqrt{\mathbb{E}[f^2(Y_t)|S_t]}} \sigma_{Dup}(t, S_t) S_t dW_t.
\]
Simulation results

- Ren, Madan and Qian (2007): solve numerically the associated Fokker-Planck PDE, and get the joint-law of \((S_t, Y_t)\).

  Subsequent extension to stochastic interest rates, stochastic dividends, multidimensional local correlation models, ...

- However, calibration errors seem to appear when the range of \(f(Y)\) is too large.
Theoretical results

- Abergel and Tachet (2010): perturbation of the constant $f$ case (Dupire) $\rightarrow$ existence for the restriction to a compact spatial domain of the associated Fokker-Planck equation when $\sup f - \inf f$ small.

- Global existence and uniqueness to LSV models remain on open problem.
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The studied problem

A simpler SDE

- The stochastic factor $Y$ is a r.v. (does not evolve with time) with values in $\mathcal{Y} := \{y_1, ..., y_d\}$ such that
  \[
  \forall i \in \{1, ..., d\}, \quad \alpha_i = \mathbb{P}(Y = y_i) > 0.
  \]
  It is independent from $S_0$ and $(W_t)_{t \geq 0}$.
- We suppose $\sigma_{Dup} \equiv 1$, $r = \frac{1}{2}$ and consider $X_t = \ln(S_t)$.
- Since $\mathbb{E}[f^2(Y) | S_t] = \mathbb{E}[f^2(Y) | X_t]$ and neglecting the drift term $\frac{1}{2} - \frac{1}{2} \frac{f^2(Y)}{\mathbb{E}[f^2(Y) | X_t]}$ with vanishing conditional expectation given $X_t$, we get

\[
\begin{cases}
  dX_t = \frac{f(Y)}{\sqrt{\mathbb{E}[f^2(Y) | X_t]}} dW_t \\
  X_0 \sim \mu_0.
\end{cases}
\]

with $X_0$, $Y$ and $(W_t)_{t \geq 0}$ independent.
- If $X_0 = 0$, we expect that $X_t \sim \mathcal{N}_1(0, t)$. 
Fake Brownian motion

**Definition**
A fake Brownian motion is a martingale \((M_t)_{t \geq 0}\) such that for all \(t \geq 0\), \(M_t \sim \mathcal{N}_1(0, t)\) and \((M_t)_{t \geq 0}\) is not a Brownian motion.


**Continuous fake Brownian motions**: Albin (2010), Oleszkiewicz (2010), Baker, Donati-Martin and Yor (2011),...

**Fake exponential Brownian motion**: Hobson (2013)
A new continuous fake Brownian motion

Lemma

If the positive function $f$ is not constant on $\mathcal{Y}$, then any solution to the SDE

$$dX_t = \frac{f(Y)}{\sqrt{\mathbb{E}[f^2(Y) | X_t]}} dW_t, \; X_0 = 0$$

with $Y$ and $(W_t)_{t \geq 0}$ indep. is a continuous fake Brownian motion.

- By Gyongy’s theorem, $\forall t \geq 0$, $X_t \sim \mathcal{N}_1(0, t)$.
- If $(X_t)_{t \geq 0}$ is a Brownian motion then a.s. $\forall t \geq 0$,
  $<X>_t = t$ i.e. $ds$ a.e. $\frac{f^2(Y)}{\mathbb{E}[f^2(Y) | X_s]} = 1 = \frac{f(Y)}{\sqrt{\mathbb{E}[f^2(Y) | X_s]}}$ so

  that a.s. $\forall t \geq 0$, $X_t = W_t$. Therefore $X_t \perp Y$,
  $\mathbb{E}[f^2(Y) | X_t] = \mathbb{E}[f^2(Y)]$ and $f^2(Y) = \mathbb{E}[f^2(Y)]$ is constant.
Existence to SDE (FBM) and fake Brownian motion

We define for $i \in \{1, \ldots, d\}$, $\lambda_i := f^2(y_i)$.

**Theorem**

*Condition (C):* there exists a symmetric positive definite $\Gamma \in \mathbb{R}^{d \times d}$ such that for all $k \in \{1, \ldots, d\}$, the $d \times d$ matrix

$$
\Gamma^{(k)}_{ij} = \frac{\lambda_i + \lambda_j}{2} (\Gamma_{ij} + \Gamma_{kk} - \Gamma_{ik} - \Gamma_{jk})
$$

is positive definite on $e_k^\perp$.

*Under (C), there exists a weak solution to the SDE (FBM).*

If $d = 2$, (C) is satisfied: choice $\Gamma = I_2$,
if $d = 3$, (C) $\iff \frac{1}{\beta_1\beta_2} + \frac{1}{\beta_2\beta_3} + \frac{1}{\beta_3\beta_1} > \frac{1}{4}$ with
$$\beta_1 = \sqrt{\frac{\lambda_2}{\lambda_3}} - \sqrt{\frac{\lambda_3}{\lambda_2}}, \beta_2 = \sqrt{\frac{\lambda_3}{\lambda_1}} - \sqrt{\frac{\lambda_1}{\lambda_3}}, \beta_3 = \sqrt{\frac{\lambda_1}{\lambda_2}} - \sqrt{\frac{\lambda_2}{\lambda_1}}$$

if $d \geq 4$, $\max_{1 \leq k \leq d} \sum_{i \neq k} \lambda_i \sum_{i \neq k} \frac{1}{\lambda_i} \leq (d + 1)^2 \Rightarrow (C)$: $\Gamma = I_d$.

Where does (C) appear?
Main Result

The associated Fokker-Planck system

- For $i \in \{1, \ldots, d\}$, define $p_i$ s.t., for $\phi \geq 0$ and measurable,
  $$\mathbb{E} \left[ \phi(X_t) 1_{\{Y=y_i\}} \right] = \int_{\mathbb{R}} \phi(x) p_i(t, x) dx.$$  

- The associated Fokker-Planck system is:

  $$\forall i \in \{1, \ldots, d\}, \frac{\partial_p t}{p_i} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \left( \frac{\sum_j p_j}{\sum_j \lambda_j p_j} \lambda_i p_i \right)$$

  $$p_i(0) = \alpha_i \mu_0$$

- $\sum_j p_j$ is solution to the heat equation.
Rewriting into divergence form

The system satisfied by \( p = (p_1, \ldots, p_d) \) can be rewritten in divergence form:

\[
\partial_t p = \frac{1}{2} \partial_x \left( \sum_{k=1}^{d} w_k(p) M^{(k)}(p) \partial_x p \right)
\]

where \( w_k(p) := \frac{\lambda_k p_k}{\sum_j \lambda_j p_j} \), \( \sum_{k=1}^{d} w_k(p) = 1 \) and for \( \overline{\lambda}(p) := \frac{\sum_j \lambda_j p_j}{\sum_j p_j} \),

\[
M^k(p) := \begin{pmatrix}
\frac{1}{\lambda} \\
\frac{\lambda_1}{\lambda} \\
\vdots \\
(1 - \frac{\lambda_1}{\lambda}) \cdot (1 - \frac{\lambda_{k-1}}{\lambda}) \cdot 1 \cdot (1 - \frac{\lambda_{k+1}}{\lambda}) \cdot \frac{\lambda_k + 1}{\lambda} \\
\frac{\lambda_{k+1}}{\lambda} \\
\vdots \\
\frac{\lambda_d}{\lambda}
\end{pmatrix}
\]

← row \( k \).
Computing Energy Estimates

- Multiply the system by $p^*(J_d + \epsilon \Gamma)$ where $(J_d)_{ij} = 1$ and $\epsilon > 0$, and integrate in $x \in \mathbb{R}$ by parts:

$$\frac{1}{2} \partial_t \left( \int_{\mathbb{R}} p^*(J_d + \epsilon \Gamma) p dx \right) = -\frac{1}{2} \int_{\mathbb{R}} \sum_{k=1}^{d} w_k(p) \partial_x p^*(J_d + \epsilon \Gamma) M^{(k)}(p) \partial_x p dx.$$

- By (C), $\exists \eta > 0$, $\forall y = (y_1, \ldots, y_d) \in \mathbb{R}^d$ with $\sum_{j=1}^{d} y_j = 0$,

$$\bar{\lambda}(p) y^* \Gamma M^{(k)}(p) y = (y - y_k e_k)^* \Gamma^{(k)}(y - y_k e_k) \geq \eta |y|^2.$$

- Since $J_d M^{(k)}(p) = J_d$ so that $y^* J_d M^{(k)}(p) y = (\sum_{i=1}^{d} y_i)^2$, we deduce that for $\epsilon$ small enough,

$$\inf_{p, y \in \mathbb{R}^d} \frac{1}{|y|^2} \sum_{k=1}^{d} w_k(p) y^* (J_d + \epsilon \Gamma) M^{(k)}(p) y > 0.$$

- $\Rightarrow$ EE in $L^2([0, T], (H^1(\mathbb{R}))^d) \cap L^\infty([0, T], (L^2(\mathbb{R}))^d)$. 
Processes matching given marginals

A new fake Brownian motion

Existence of Calibrated RSLV models

Ideas of proof

Step 1/3: Existence to an approximate PDS when

\[ \mu_0(dx) = p_0(x)dx, \ p_0 \in L^2(\mathbb{R}) \]

For \( \epsilon > 0 \), use Galerkin’s method to solve an approximate PDE:

\[
\frac{\partial_t p^\epsilon}{2} = \frac{\partial_x}{2} (M^\epsilon(p^\epsilon) \partial_x p^\epsilon) \quad p^\epsilon(0) = (\alpha_1, \ldots, \alpha_d) p_0.
\]

where

\[
\sum_{k=1}^d w_k(\rho) M_{ii}^{(k)}(\rho) = 1 + \frac{\sum_{l \neq i} \lambda_l \rho_l \sum_l (\lambda_i - \lambda_l) \rho_l}{(\sum_l \lambda_l \rho_l)^2},
\]

\[
M_{ii}^\epsilon(\rho) = 1 + \frac{\sum_{l \neq i} \lambda_l \rho_l^+ \sum_l (\lambda_i - \lambda_l) \rho_l^+}{(\epsilon \lor \sum_l \lambda_l \rho_l^+)^2},
\]

and for \( j \neq i \),

\[
\sum_{k=1}^d w_k(\rho) M^{(k)}(\rho)_{ij} = \frac{\lambda_i \rho_i \sum_l (\lambda_l - \lambda_j) \rho_l}{(\sum_l \lambda_l \rho_l)^2},
\]

\[
M_{ij}^\epsilon(\rho) = \frac{\lambda_i \rho_i^+ \sum_l (\lambda_l - \lambda_j) \rho_l^+}{(\epsilon \lor \sum_l \lambda_l \rho_l^+)^2}.
\]
Step 1/3: Existence to an approximate PDS when \( \mu_0 \in L^2(\mathbb{R}) \)

- \( \rho \mapsto M^\epsilon(\rho) \) locally Lipschitz and bounded \( \rightarrow \exists! \) solution \( p^\epsilon_m \) to a projection of the equation in dimension \( m \).
- Coercivity uniform in \( \epsilon \) under (C): \( \exists \) solution \( p^\epsilon \) satisfying uniform in \( \epsilon \) EE by taking the limit \( m \rightarrow \infty \).
- Taking \( p^\epsilon_- \) as test function, we show that \( p^\epsilon \geq 0 \).
- \( \forall \epsilon, \forall i, \sum_j M^\epsilon_{ji} = 1 \implies \sum_j p^\epsilon_j \) solves the heat equation \( \rightarrow \) lower bound uniform in \( \epsilon \) (but not \( t, x \)) for \( \sum_j \lambda_j p^\epsilon_j \).
- \( \epsilon \rightarrow 0 \), existence of a solution to the original PDS.
Step 2/3: Existence to the PDS when $\mu_0 \in \mathcal{P}(\mathbb{R})$

- By mollification of $\mu_0$, we use the results of Step 1 to extract a solution to the PDS when $\mu_0 \in \mathcal{P}(\mathbb{R})$.

- We use the fact that $\sum_j p_j$ is solution to the heat equation to control the rate of explosion of $t \mapsto \int_{\mathbb{R}} \sum_{i=1}^{d} p_i^2(t, x) \, dx$ as $t \to 0$ uniformly in the mollification parameter.
Step 3/3: Existence of a weak the SDE (FBM)

Theorem (Figalli (2008))

For \(a: [0, T] \times \mathbb{R} \to \mathbb{R}_+\) and \(b: [0, T] \times \mathbb{R} \to \mathbb{R}\) meas. and bounded let \(L_t \varphi(x) = \frac{1}{2} a(t, x) \varphi''(x) + b(t, x) \varphi'(x)\).

If \([0, T] \ni t \mapsto \mu_t \in \mathcal{M}_+(\mathbb{R})\) is weakly continuous and solves the Fokker-Planck equation \(\partial_t \mu_t = L^* \mu_t\) in the sense of distributions then there exists a probability measure \(P\) on \(C([0, T], \mathbb{R})\) with marginals \((P_t = \mu_t)_{t \in [0, T]}\) such that
\[
\forall \varphi \in C^2_b(\mathbb{R}), \quad \varphi(X_t) - \int_0^t L_s \varphi(X_s) \, ds \text{ is a } P\text{-martingale.}
\]

\[\Rightarrow \text{for } i \in \{1, \ldots, d\}, \text{ there exists a probab. } P^i \text{ on } C([0, T], \mathbb{R}) \text{ with } P^i_0 = \mu_0 \text{ and } P^i_t = \frac{p_i(t, x) \, dx}{\alpha_i} \text{ for } t \in (0, T] \text{ and } \forall \varphi \in C^2_b(\mathbb{R}), \]
\[
\varphi(X_t) - \int_0^t \frac{f^2(y_i) \sum_{j=1}^d p_j}{\sum_{j=1}^d f^2(y_j) p_j} (s, X_s) \varphi''(X_s) \, ds \text{ is a } P^i\text{-martingale.}
\]

\[P(dX, dY) = \sum_{i=1}^d \alpha_i P^i(dX) \otimes \delta_{y_i}(dY)\] weak solution.
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The calibrated RSLV model

Presentation

- We consider the following dynamics (RSLV):

\[
dS_t = rS_t dt + \frac{f(Y_t)}{\sqrt{\mathbb{E}[f^2(Y_t)|S_t]}} \sigma_{Dup}(t, S_t) S_t dW_t,
\]

where \((Y_t)_{t \geq 0}\) takes values in \(\mathcal{Y}\), and

\[
\mathbb{P}(Y_{t+dt} = y_j | Y_t = y_i, S_t = x) = q_{ij}(x) dt.
\]

- **Switching** diffusion, special case of LSV model.

- Jump distributions and intensities are functions of the asset level.
The calibrated RSLV model

Assumptions

- (C), (Coerc. 1): $f$ satisfies condition (C).
- (HQ), (Bounded I) $\exists \bar{q} > 0$, s.t. $\forall x \in \mathbb{R}, |q_{ij}(x)| \leq \bar{q}$.

We define $\tilde{\sigma}_{Dup}(t, x) := \sigma_{Dup}(t, e^x)$.

- (H1), (Bounded vol.) $\tilde{\sigma}_{Dup} \in L^\infty([0, T], W^{1,\infty}(\mathbb{R}))$.
- (H2), (Coerc. 2) $\exists \sigma > 0$ s.t. $\sigma \leq \tilde{\sigma}_{Dup}$ a.e. on $[0, T] \times \mathbb{R}$.
- (H3), (Regul. 1) $\exists \eta \in (0, 1], \exists H_0 > 0$, s.t. $\forall s, t \in [0, T], \forall x, y \in \mathbb{R}$,

\[ |\tilde{\sigma}_{Dup}(s, x) - \tilde{\sigma}_{Dup}(t, y)| \leq H_0 (|x - y|^{\eta} + |t - s|^{\eta}) . \]

(HQ), (H1) and (H2) permit to generalize the energy estimations to the Fokker-Planck system associated with $((\ln(S_t), Y_t))_{t\in[0, T]}$.

With (H3), uniqueness and Aronson estimates for the Fokker-Planck equation associated with $(\ln(S_{t,Dup}^{Dup}))_{t\in[0, T]}$ where

\[ dS_{t,Dup}^{Dup} = \sigma_{Dup}(t, S_{t,Dup}^{Dup})S_{t,Dup}^{Dup} dW_t + rS_{t,Dup}^{Dup} dt, \quad S_{0,Dup}^{Dup} = S_0. \]

$\rightarrow$ replaces the heat equation.
Main results

Theorem

Under Conditions (H1)-(H3), (HQ) and (C) there exists a weak solution to the SDE (RSLV). Moreover, it has the same marginals as the solution to the local volatility SDE

\[
\begin{align*}
    dS_t^{Dup} &= \sigma_{Dup}(t, S_t^{Dup}) S_t^{Dup} dW_t + rS_t^{Dup} dt, \\
    S_0^{Dup} &= S_0.
\end{align*}
\]

- We generalize the results of Figalli to the regime switching case.
- Uniqueness?
Main results

A new fake Brownian motion

Existence of Calibrated RSLV models

Processes matching given marginals

Main obstacle to deal with usual LSV models

Corresponding generalized (FBM) equation

\[
\begin{align*}
    dX_t &= \frac{f(Y_t)}{\sqrt{\mathbb{E}[f^2(Y_t)|X_t]}} dW_t \\
    dY_t &= \eta(Y_t)(\rho dW_t + \sqrt{1 - \rho^2} dB_t) + b(Y_t) dt
\end{align*}
\]

with \((B_t)_{t \geq 0}\) Brownian motion indep. of \((W_t)_{t \geq 0}\).

Typically \(|\rho| \neq 1\) and \((X_t, Y_t)\) should admit a density \(p(t, x, y)\).

Even when \(0 < \inf f \leq \sup f < \infty\), in the divergence form of the term

\[
\partial_{xx} \left( \frac{f^2(y) \int p(t, x, z) dz}{\int f^2(z)p(t, x, z) dz} p(t, x, y) \right) = \partial_x \left( \frac{f^2(y) \int p(t, x, z) dz}{\int f^2(z)p(t, x, z) dz} \partial_x p(t, x, y) \right)
\]

\[
+ \partial_x \left( \frac{f^2(y)p(t, x, y)}{\int f^2(z)p(t, x, z) dz} \int \partial_x p(t, x, z) dz - \frac{f^2(y)p(t, x, y) \int p(t, x, z) dz}{(\int f^2(z)p(t, x, z) dz)^2} \int f^2(z) \partial_x p(t, x, z) dz \right)
\]

the red factors replacing \(\frac{f^2(y_i)p_i(t, x)}{\sum_{j=1}^d f^2(y_j) p_j(t, x)}\) and \(\frac{f^2(y_i)p_i(t, x) \sum_{j=1}^d p_j(t, x)}{(\sum_{j=1}^d f^2(y_j)p_j(t, x))^2}\)

are no longer bounded (same problem for kernel approximations of \(\mathbb{E}[f^2(Y_t)|X_t]\)).
Thank you for your attention!