Pricing CVA adjustments: An expansion approach for WWR

Marouan Iben Taarit

Natixis & Cermics (Paris, France)

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Introduction

- **XVA** is one of the most demanding issues in terms of prices and Greeks calculations
  - global portfolio pricing and collateral netting
  - incremental charging and reallocation
  - management of cross-asset and hybrid risks (eg. *WWR*).
- Consistency with spot prices is required (reusing validated and proven pricers)
- We show in Iben Taarit (2015) how to upgrade existant pricers in order to account for *WWR*
A brief reminder on XVA (cont’)

- At valuation time 0, we define the Bilateral Credit Valuation Adjustment (BCVA) as seen by the bank $B$ as

$$BCVA(0) = LGD_C \mathbb{E} \left[ 1_{\{\tau_C \leq \tau_B\}} 1_{\{\tau_C < T\}} D(0, \tau_C)(NPV(\tau_C, T))^+ \right]$$  \hspace{1cm} (1)

- Analog formula for the Bilateral Debt Valuation Adjustment (DVA)
- A classical simplifying assumption consists in considering the default of only one counterparty

$$UCVA(0, T) = LGD_C \mathbb{E} \left[ 1_{\{\tau_C < T\}} D(0, \tau_C)(NPV(\tau_C, T))^+ \right]$$ \hspace{1cm} (2)

- Unilateral adjustments = mutual exclusion of defaults
Pricing framework

- **Main goal**: approximated price (fast/accurate) of a contingent claim \( h(X_T) \) subject to the default of the supplying party, i.e. \( \tau < T \)
- Stochastic intensity model for the default time \( \tau \)

\[
\begin{align*}
\tau^e &= \inf \left\{ t > 0 \mid \int_0^t \lambda^e_s ds > \xi \right\} \\
d\lambda^e_t &= \kappa(t) (\psi(t) - \lambda^e_t) \, dt + \epsilon \nu(t, \lambda^e_t) \, dW_t \\
\lambda^e_0 &= \lambda_0 > 0
\end{align*}
\] 

where \( W : \mathbb{R} \)-valued SBM, \( \xi \xrightarrow{\mathcal{L}} \mathcal{E}(1), \epsilon \in [0, 1] \)

- In addition, let \((X_t)_{t \geq 0}\) be a \( \mathbb{R}^n \)-valued diffusion process governed by

\[
dX_t = (\Phi(t) + \Theta(t) X_t) \, dt + \Sigma(t) \, dB_t \, , \, X_0 \in \mathbb{R}^n
\]

with \( \Phi : [0, T] \rightarrow \mathbb{R}^n \), \( \Theta : [0, T] \rightarrow \mathbb{R}^{n \times n} \), \( \Sigma : [0, T] \rightarrow \mathbb{R}^{n \times d} \) and \((B_t)_{t \geq 0}\) a \( \mathbb{R}^d \)-valued SBM.
We define the instantaneous correlations \( \rho = (\rho_i)_{i=1 \ldots d} \) such that

\[
d \langle W, B_i \rangle_t = \rho_i dt , \quad 1 \leq i \leq d
\]  

(5)

\( \rho \neq 0 \) and \( \nu(t, \lambda_t^\epsilon) \neq 0 \) \( \Rightarrow \) Wrong-way risk

Recovery in market value convention (recovery rate \( \delta \))

\[
u_{h,\delta}^\epsilon (0, T) = \mathbb{E} \left[ h(X_T) 1_{\{\tau > T\}} + (1 - \delta) u_{h,\delta}^\epsilon (\tau^-, T) 1_{\{\tau \leq T\}} \right] \quad (6)
\]

Duffie and Singleton (1999)

\[
u_{h,\delta}^\epsilon (0, T) = \mathbb{E} \left[ \exp \left( - (1 - \delta) \int_0^T \lambda_t^\epsilon dt \right) h(X_T) \right] \quad (7)
\]
Pricing methodology

Main Objective

$$u_{h,\delta}^\epsilon (S, T) = u_{h,\delta}^\epsilon (S, T) + \text{weighted sum of Greeks of } \mathbb{E} [h (X_T)] + \text{Error}$$

where

- $$u_{h,\delta}^0 (T) = e^{-(1-\delta) \int_0^T \lambda_t^0 dt} \mathbb{E} [h (X_T)]$$ (classical pricing)
- The weighted sum of Greeks of $$\mathbb{E} [h (X_T)]$$ is given by the system

we want the accuracy to be bounded using

- The regularity of the intensity process $$\lambda_t^\epsilon$$
- The dependence of $$u_{h,\delta}^\epsilon$$ on $$\int \lambda_s^\epsilon ds$$
Comparison with similar works

- Expansion approach for credit intensity diffusion already addressed in Muroi 2005 and Muroi 2012
  - Linearization of the payoff function \( \Phi(e^r(T), e^\lambda(T)) \) (smoothness requirements)
- We follow Benhamou et al. (2009, 2010a,b). However, the setting is fundamentally different.
  - We perform expansion for \( e^{-(1-\delta) \int_0^T \lambda_s^e ds} \)
  - Minimal dependence on the regularity of \( h \)
Theorem (Second order approximation)

Under regularity assumption of the drift and diffusion of \((\lambda^\varepsilon_t)\), one has

\[
\begin{align*}
    u_{h,\delta}^{\varepsilon=1} (0, T) &= u_{h,\delta}^{\varepsilon=0} (0, T) + (1 - \delta)^2 \; C_{0,1} (T) u_{h,\delta}^{\varepsilon=0} (0, T) \\
    &\quad \quad - (1 - \delta) \; C_{1,1} (T) \cdot \text{Greek}^{(1)} (T, X_T) \\
    &\quad \quad - (1 - \delta) \left( C_{2,1} (T) - (1 - \delta) \; C_{2,2} (T) \right) \cdot \text{Greek}^{(2)} (T, X_T) \\
    &\quad \quad + \text{Error}^{\varepsilon=1}_2
\end{align*}
\]

- \( C_{0,1} (T) = \frac{1}{2} \int_0^T \left( \int_t^T e^{-\int_s^t \kappa(u) du} ds \nu(t) \right)^2 dt \)
- \([C_{1,1} (T)]_i = \int_0^T \left( \int_t^T e^{-\int_s^t \kappa(u) du} ds \right) \nu(t) \left[ \Sigma(t, T) \rho \right]_i dt \)
- \([C_{2,1} (T)]_{i,j} = \int_0^T \left( \int_t^T \left( \int_s^T e^{-\int_u^s \kappa(v) dv} du \right) \nu^{(1)} (s) \left[ \Sigma(s, T) \rho \right]_i ds \right) \nu(t) \left[ \Sigma(t, T) \rho \right]_j dt \)
- \([C_{2,2} (T)]_{i,j} = \ldots \)
Numerical experiments

- Log-normal diffusion of the spot $S_t$. Default parameters are

<table>
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<tr>
<th>$T$</th>
<th>$r/q/d$</th>
<th>$\Sigma$</th>
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- 3 models of $\lambda_t^e$

<table>
<thead>
<tr>
<th>$\mathcal{N}$</th>
<th>$\mathcal{LN}$</th>
<th>$\mathcal{C}$</th>
</tr>
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<tbody>
<tr>
<td>$\nu(t, \lambda_t^e) = \bar{\nu}_n$</td>
<td>$\nu(t, \lambda_t^e) = \bar{\nu}_{ln}\lambda_t^e$</td>
<td>$\nu(t, \lambda_t^e) = \bar{\nu}_c\sqrt{\lambda_t^e}$</td>
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- 2 risk regimes

<table>
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<tr>
<th>Mid Risk</th>
<th>$\lambda_0$</th>
<th>$\kappa$</th>
<th>$\psi$</th>
<th>$\rho$</th>
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<table>
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<th>High Risk</th>
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<th>$\psi$</th>
<th>$\rho$</th>
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<td>3%</td>
<td>100%</td>
<td>50%</td>
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- Benchmarking *versus* Monte Carlo (Paths = 10^5, 24 steps/ year)
### Contingent Call option

#### (a) Relative Error: Mid risk parameters

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<thead>
<tr>
<th></th>
<th>K/S</th>
<th>CI</th>
<th>Proxy</th>
<th>2nd Order Exp.</th>
<th>3rd Order Exp.</th>
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#### (b) Relative Error: High risk parameters

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<td>54.05%</td>
<td>9.76%</td>
<td>5.41%</td>
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</table>
A Wrong-way risk adjustment for CVA/DVA

- UCVA is usually approximated by

\[
UCVA(0) \approx LGD_C \sum_{m=1}^{M} D(0, T_m) \mathbb{E} \left[ (1_{T_{m-1} < \tau_C} - 1_{T_m < \tau_C}) (V(T_m, X_{T_m}))^+ \right]
\]

\[
\approx LGD_C \sum_{m=1}^{M} D(0, T_m) (u^e_{V+}(T_{m-1}, T_m) - u^e_{V+}(T_m, T_m))
\]

where

\[
u^e_{V+},0 : (s, t) \mapsto \mathbb{E} \left[ e^{-\int_s^t \lambda^e_{\omega} d\omega} (V(t, X_t))^+ \right]
\]

Consequence

With \( \delta = 0 \) and \( h = V^+ \), our approximation formulas yield

\[
UCVA(0) = UCVA^0(0) + \sum_{m=1}^{M} \text{weighted sum of Greeks of } \mathbb{E} \left[ (V(X_{T_m}))^+ \right] + \text{Error}
\]

Wrong-way Risk Adjustment = \( UCVA(0) - UCVA^0(0) \)

= Weighted sum of exposure Greeks
A WWR adjustment in the bilateral framework
We apply the same methodology

\[ BCVA(0) \]
\[ \approx LGD_C \sum_{m=0}^{M} D(0, T_m) \mathbb{E} \left[ \left( 1_{\{ \tau_C \geq T_{m-1} \}} - 1_{\{ \tau_C \geq T_m \}} \right) 1_{\{ \tau_B > T_m \}} (V(T_m, X_{T_m}))^+ \right] \]
\[ \approx LGD_C \sum_{m=1}^{M} D(0, T_m) \left( u^{\epsilon}_{V+,0}(T_{m-1}, T_m) - u^{\epsilon}_{V+,0}(T_m, T_m) \right) \]

where

\[ u^{\epsilon}_{V+,0}(s, t) = \mathbb{E} \left[ 1_{\{ \tau_C \geq s \}} 1_{\{ \tau_B \geq t \}} (V(t, X_t))^+ \right] \]
\[ = \mathbb{E} \left[ \left( e^{-\int_0^s \lambda_C \omega \, d\omega} e^{-\int_0^t \lambda_B \omega \, d\omega} \right) (V(t, X_t))^+ \right] \]

Consequence
With \( \delta = 0 \) and \( h = V^+ \), our approximation formulas yield

\[ BCVA(0) = UCVA^0(0) + \sum_{m=1}^{M} \text{weighted sum of Greeks of } \mathbb{E} \left[ (V(X_{T_m}))^+ \right] + \text{Error} \]

\[ \text{Bilateral Wrong-way Risk} = UWWR_C + UWWR_B + \text{First to default Risk} \]
References


