Interest rate models enhanced with local volatility

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Introduction

Background:

- In fixed income world, Dupire-like local volatility does not exist.

- In [3], Gatarek et al have considered a one-dimensional Cheyette process enhanced with a local vol, and derived an (approximate) Dupire-like local vol.

- Dimensional curse for markov functionals.
Introduction

Our work:

- A general equation which impose a generic (multi-factors) interest rate model is calibrated to a strip of rolling maturity swaptions. (Constant tenor, diagonals, etc.)

- As an example: Cheyette 1-d enhanced with local vol.

- Extension to multi-dimensional models (Cheyette, BGM).
Matching a rolling maturity swaption: constant tenor

Rolling Swap rate $s_{t}^{t,t+\theta}$ with maturity $t$ and tenor $\theta$, its dynamics is:

$$
\begin{align*}
\frac{ds_{t}^{t,t+\theta}}{dt} &= \sigma_{t}^{t,t+\theta} dW_{t}^{t,t+\theta} + \cdot dt
\end{align*}
$$

\text{target local vol}

**Proposition (Local vol calibration condition)**

$$
C(t, K) = C^{\text{mkt}}(t, K) \text{ for all } (t, K) \text{ if and only if}
$$

$$
E^{Q}\left[ (\sigma_{t}^{t,t+\theta})^2 | s_{t}^{t,t+\theta} = K \right] = 2 \frac{\partial_{t}C^{\text{mkt}}(t,K) - KC^{\text{mkt}}(t,K) + K^2 \partial_{K}C^{\text{mkt}}(t,K)}{\partial_{K}^2C^{\text{mkt}}(t,K)}
$$

$$
+ 2 \frac{E^{Q}\left[ \frac{1}{B_{t}} s_{t}^{t,t+\theta} > K (f_{t,t} - f_{t,t+\theta} P_{t,t+\theta}) \right]}{\partial_{K}^2C^{\text{mkt}}(t,K)}
$$

(1)

$f_{t,\alpha}$: forward instantaneous rate at $\alpha$.
$P_{t,\alpha}$: zero coupon of maturity $\alpha$. 

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An example: Cheyette’s model with LV

- Markovian processes:

\[ dX_t = (Y_t - \lambda_t X_t) \, dt + \sigma_t \, dW_t \]
\[ dY_t = (\sigma_t^2 - 2\lambda Y_t) \, dt \]

- zero-coupon bond:

\[ P_{tT} = \frac{P_{0T}}{P_{0t}} \, e^{G_{tT} X_t - \frac{1}{2} G_{tT}^2 Y_t}, \quad G_{tT} = \frac{e^{-\lambda(T-t)} - 1}{\lambda} \]
\[ f_{tT} \equiv -\partial_T \ln P_{tT} = f_{0T} + e^{-\lambda(T-t)} \left( X_t - G_{tT} Y_t \right) \]
\[ r_t = f_{0t} + X_t \]

- Local volatility specification: \( \sigma_t = \frac{\sigma(t, s_{t,t+\theta(t)})}{(\partial_X s_{t,t+\theta(t)})(t, X_t, Y_t)} \)

- Swap rate dynamics is then:

\[ ds_{t,t+\theta} = \partial_X s_{t,t+\theta(t)}(t, X_t, Y_t) \cdot dX_t = \sigma(t, s_{t,t+\theta(t)}) \cdot dW_{t,t+\theta} \]
Matching marginals

From Equation (1), \( C(t, K) \equiv C_{\text{mkt}}(t, K) \) for all \((t, K) \in [0, T] \times \mathbb{R}\) if and only if \( \sigma(t, K) \) is given by

\[
\sigma(t, K)^2 = \sigma_{\text{loc}}(t, K)^2 + 2 \frac{\Xi(t, K)}{\partial_k^2 C_{\text{mkt}}(t, K)}
\]

with

\[
\Xi(t, K) \equiv \mathbb{E}^Q[e^{-\int_0^t r_s ds} \xi_t]
\]

\[
\xi_t \equiv 1_{s_t^{t+\theta} > K} (f_{t,t} - f_{t,t+\theta} P_{t,t+\theta})
\]

Numerical solutions \(\Rightarrow\) particle method:

\(\Rightarrow\) Approximate \(\Xi(t, K)\) by empirical distribution at each step
Numerical examples

**Figure:** Swaption smile with maturities 5Y/10Y and tenor of 5Y compared to implied volatilities (EUR, 15-April-2016). $N = 2^{12}$ particles, $2^{15}$ simulations.
**Figure:** Swaption smile with maturities 5Y/10Y and tenor of 10Y compared to implied volatilities (EUR, 15-April-2016). $N = 2^{12}$ particles, $2^{15}$ simulations.
Extensions to multi-dimensional Cheyette

- A multi-dimensional Cheyette model:
  \[ dx_t = \cdot dt + \Sigma(t, x_t) \cdot dW_t \]
  \( \Sigma(t, x_t) \) is \( N \times N \), \( W_t \) a \( N \)-dimensional Brownian motion.

- Swap rate dynamics:
  \[ ds_t^{\alpha, \beta} = \left( \nabla_x s_t^{t, t+\theta} \right) \Sigma(t, x_t) \cdot dW_t^{\alpha, \beta} \]

- Take \( \Sigma(t, x_t) = \left( \nabla_x s_t^{t, t+\theta} \right)^{-1} \sigma(t, s_t^{t, t+\theta}) \Phi(t) \) where \( \sigma(t, s_t^{t, t+\theta}) \) is a scaling function and \( \Phi(t) \) a deterministic \( N \times N \) matrix.

- Calibration condition is:
  \[
  \sigma(t, K)^2 \text{Tr} \left( \Phi(t)^\dagger \cdot \Phi(t) \right) = \sigma_{\text{loc}}(t, K)^2 + 2 \frac{\Xi(t, K)}{\partial K^2} C_{\text{mkt}}(t, K)
  \]
Extensions to Libor market models

- Swap rate dynamics under LLM:

\[ ds_t^{\alpha,\beta} = \sum_{i=\alpha+1}^{\beta} \frac{\partial s_t^{\alpha,\beta}}{\partial L_t^i} \Sigma_t^i \cdot dW_t^{\alpha,\beta}, \]

\( \Sigma_t^i \) is the instantaneous volatility of Libor \( L_t^i \equiv L(t, T_i, T_{i+1}) \).

- Take

\[ \Sigma_t^i = \left( \frac{\partial s_t^{t,t+\theta}}{\partial L_t^i} \right)^{-1} \sigma(t, s_t^{t,t+\theta}) \Phi_i(t) \]
Open questions

- Existence of local vol: existence of solution of McKean-Vlasov type SDE.
- Calibration on Vol cube: how to parametrize?


