Branching diffusion representation for semilinear PDEs and Monte-Carlo approximation

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Outline

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2 Branching diffusion representation for semilinear PDEs
   • A first class of semilinear PDEs
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   • Drawback and extensions
1 Introduction

2 Branching diffusion representation for semilinear PDEs
   • A first class of semilinear PDEs
   • A second class of semilinear PDEs
   • Drawback and extensions
Feynmann-Kac formula

- Let \( W \) be a standard \( d \)-dimensional Brownian motion, \( f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R} \) and \( g : \mathbb{R}^d \rightarrow \mathbb{R} \) be continuous and of polynomial growth.

- The heat equation on \([0, T] \times \mathbb{R}^d\):

\[
(\partial_t u + \frac{1}{2} \Delta u + f)(t, x) = 0, \quad u(T, \cdot) = g(\cdot). \tag{1}
\]

- A probabilistic representation

\[
u(t, x) = \mathbb{E} \left[ g(x + W_{T-t}) + \int_t^T f(s, x + W_{s-t}) ds \right]. \tag{2}
\]

Theorem (Feynmann-Kac et the reverse)

(i) Let \( u \in C^{1,2}([0, T] \times \mathbb{R}^d) \) be a classical solution of PDE (1), then \( u \) has the probabilistic representation (2).

(ii) Let the function \( u(t, x) \) be defined by (2), then \( u \) is a “solution” of PDE (1).
The semilinear parabolic PDE:

$$\partial_t u(t, x) + \frac{1}{2} \Delta u(t, x) + f(\cdot, u, Du)(t, x) = 0, \quad u(T, \cdot) = g(\cdot).$$

The backward SDE:

$$Y_t = g(W_T) + \int_t^T f(s, W_s, Y_s, Z_s)ds + \int_t^T Z_s dW_s.$$ 

Theorem

(i) Let $u \in C^{1,2}([0, T] \times \mathbb{R}^d)$ be a solution of the semilinear PDE, then $(Y_t, Z_t) := (u(t, W_t), Du(t, W_t))$ provides a solution of the BSDE.

(ii) Define $u(t, x) := Y_t^{t, x}$, then $u$ is a “solution” of semilinear PDE.
Introduction
Branching diffusion representation for semilinear PDEs

Branching diffusion process

Nonlinearity:
\( \beta(\alpha v^2 + \alpha v - v) \)
Branching diffusion process and semi-linear PDE

\[ \mathcal{K}_t := \{ \text{All particles alive at time } t \} , \quad \overline{\mathcal{K}}_T := \bigcup_{t \leq T} \mathcal{K}_t. \]

- [Skorokhod, Watanebe, McKean, etc.] Representation of KPP equation

\[ \partial_t u + \frac{1}{2} \Delta u + \beta \left( \sum_{\ell \in \mathbb{N}} p_\ell u^\ell - u \right) = 0, \quad u(T, \cdot) = g(\cdot), \]

by branching Brownian motion \( \mathbb{E} \left[ \prod_{k \in \mathcal{K}_T} g(W^k_T) \right]. \)


\[ \partial_t u + \frac{1}{2} \Delta u + \beta \left( \sum_{\ell \in \mathbb{N}} p_\ell a_\ell u^\ell - u \right) = 0, \quad u(T, \cdot) = g(\cdot), \]

by \( \mathbb{E} \left[ \left( \prod_{k \in \mathcal{K}_T \setminus \mathcal{K}_T} a_k(T_k, W^k_T) \right) \left( \prod_{k \in \mathcal{K}_T} g(W^k_T) \right) \right]. \)
Objective

- Extension

\[
\sum_{\ell \in \mathbb{N}} p_{\ell} a_{\ell} u_{\ell} \rightarrow \sum_{\ell=(\ell_0, \ell_1, \cdots, \ell_m)} p_{\ell} a_{\ell} u_{\ell_0} \prod_{i=1}^{m} (b_i \cdot Du)^{\ell_i}.
\]

- **Forward numerical algorithm** for semilinear PDE.
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A branching Brownian motion

Nonlinearity:
$\beta(a_2 v^2 + a_1 v - v)$
A branching Brownian motion

A branching Brownian motion $(W^k)_{k \in \overline{K}_T}$:

- $\mathcal{K}_t$ the collection of particles alive at time $t$, $\overline{\mathcal{K}}_T := \cup_{t \leq T} \mathcal{K}_t$.
- $\mathcal{K}^n_t \subset \mathcal{K}_t$ (resp. $\overline{\mathcal{K}}^n_T$) the collection of particles of generation $n$.
- A particle $k \in \overline{\mathcal{K}}_T$ defaults at time $T_k \wedge T$.
- For a particle $k \in \overline{\mathcal{K}}_T$, denote by $k-$ its parent particle, then the particle starts its life at time $T_{k-}$ and ends its life at time $T_k \wedge T$.
- $\Delta T_k := T_k - T_{k-}$ follows a distribution of density $\rho$ (e.g. $\mathcal{E}(\beta)$ with $\rho(t) = \beta e^{-\beta t} 1_{t \geq 0}$), denote $\overline{F}(T) := \int_T^{\infty} \rho(s)ds$.
- At default time $T_k < T$, the particle $k$ branches into $l_k$ offspring particles, where $\mathbb{P}(l_k = \ell) = p_\ell$.
- For $k \in \overline{\mathcal{K}}_T$, $W^k_{T_k-} = W^{k-}_{T_k-}$.
A KPP equation and branching Brownian motion

- KPP equation: Let \( u \in C^{1,2} \) be a solution to

\[
\partial_t u + \frac{1}{2} \Delta u + \sum_{\ell \in \mathbb{N}} a_\ell u^\ell = 0, \quad u(T, \cdot) = g(\cdot).
\]

- Using Feynmann-Kac

\[
u(0, 0) = \mathbb{E} \left[ g(W_T) + \int_0^T \sum_{\ell} (a_\ell u^\ell)(s, W_s) ds \right]
= \mathbb{E} \left[ \frac{g(W_T)}{F(T)} F(T) + \int_0^T \sum_{\ell} (a_\ell u^\ell)(s, W_s) \frac{1}{\rho(s)} \rho(s) ds \right]
= \mathbb{E} \left[ \frac{g(W_T^{(1)})}{F(T)} \mathbb{I}\{T^{(1)} \geq T\} + \frac{a_{l^{(1)}}(T^{(1)}, W_{T^{(1)}})}{p_l(T^{(1)}) \rho(T^{(1)})} u^{l^{(1)}}_{T^{(1)}} \mathbb{I}\{T^{(1)} < T\} \right].
\]
A KPP equation and branching Brownian motion ...

- Let us introduce

\[
\psi_1 := \left[ \prod_{k \in \mathcal{K}_T^1} \frac{g(W_k^T)}{F(\Delta T_k)} \right] \left[ \prod_{k \in \mathcal{K}_T^1 \setminus \mathcal{K}_T^2} \frac{a_{l_k}(T_k, W_k^T)}{p_{l_k} \rho(T_k)} \right] \left[ \prod_{k \in \mathcal{K}_T^2} u_{T_k^-} \right].
\]

Then

\[
u(0, 0) = \mathbb{E} \left[ \frac{g(W_T^{(1)})}{F(T)} \mathbb{1}_{\{T(1) \geq T\}} + \frac{a_{l(1)}(T(1), W_{T(1)}^{(1)})}{p_{l(1)} \rho(T(1))} u_{T(1)}^{l(1)} \mathbb{1}_{\{T(1) < T\}} \right]
\]

\[= \mathbb{E} [\psi_1].\]
A KPP equation and branching Brownian motion ...

- Recall

\[ 
\psi_1 := \left[ \prod_{k \in \mathcal{K}_T^1} \frac{g(W_k^T)}{F(\Delta T_k)} \right] \left[ \prod_{k \in \overline{\mathcal{K}_T^1} \setminus \mathcal{K}_T^1} \frac{a_{l_k}(T_k, W_k^T)}{p_{l_k} \rho(T_k)} \right] \left[ \prod_{k \in \mathcal{K}_T^2} u_{T_{k-}} \right] 
\]

and

\[ u(0, 0) = \mathbb{E}[\psi_1]. \]

- By iteration, one has

\[ u(0, 0) = \mathbb{E}[\psi_n] = \mathbb{E}\left[ \lim_{n \to \infty} \psi_n \right] = \mathbb{E}[\psi], \]

where

\[ \psi_n := \left[ \prod_{k \in \bigcup_{j=1}^n \mathcal{K}_T^j} \frac{g(W_k^T)}{F(\Delta T_k)} \right] \left[ \prod_{k \in \bigcup_{j=1}^n \overline{\mathcal{K}_T^j} \setminus \mathcal{K}_T^j} \frac{a_{l_k}(T_k, W_k^T)}{p_{l_k} \rho(T_k)} \right] \left[ \prod_{k \in \overline{\mathcal{K}_T}^{n+1}} u_{T_{k-}} \right]. \]

\[ \psi := \left[ \prod_{k \in \mathcal{K}_T} \frac{g(W_k^T)}{F(\Delta T_k)} \right] \left[ \prod_{k \in \overline{\mathcal{K}_T} \setminus \mathcal{K}_T} \frac{a_{l_k}(T_k, W_k^T)}{p_{l_k} \rho(T_k)} \right]. \]
A second class of semilinear equation

• A second class of semilinear PDEs: Let $u \in C^{1,2}$ be a solution to
\[
\partial_t u + \frac{1}{2} \Delta u + \sum_{\ell=(\ell_0, \ell_1)} a_{\ell} u^{\ell_0} (b \cdot Du)^{\ell_1} = 0, \quad u(T, \cdot) = g(\cdot).
\]

• Using Feynmann-Kac (let $\mathbb{P}[l_{(1)} = \ell = (\ell_0, \ell_1)] = p_\ell$)
\[
u(0, 0) \equiv \mathbb{E} \left[ g(W_T) + \int_0^T \sum_\ell (a_{\ell} u^{\ell_0} (b \cdot Du)^{\ell_1})(s, W_s) ds \right]
\]
where
\[
\psi_1 := \left[ \prod_{k \in \mathcal{K}_T^1} \frac{g(W_{T_k})}{F(\Delta T_k)} \right] \left[ \prod_{k \in \overline{\mathcal{K}_T^1} \setminus \mathcal{K}_T^1} \frac{a_{l_k}(T_k, W_{T_k})}{p_{l_k}(T_k)} \right] \left[ \prod_{k \in \mathcal{K}_T^2} u_{T_k-} \right.
\]
Seminlinear equation and branching Brownian motion

- Modification on Branching process:
  - $\mathbb{P}[I_k = \ell = (\ell_0, \ell_1)] = p_\ell$.
  - At default time, the particle $k$ branches into $|I_k|$ (independent) particles, among which $I_{k,0}$ particles are marked by 0, and $I_{k,1}$ particles are marked by 1.
  - Denote by $\theta_k$ the mark of $k$ (initial particle is marked by 0).

- Automatic differentiation, let $v(x) = \mathbb{E}[\phi(x + \Delta W)]$, then by integral by part formula,

$$Dv(x) = \mathbb{E} \left[ \phi(x + \Delta W) \frac{\Delta W}{\Delta T} \right]$$

- Then

$$(b \cdot Du)(0,0) = \mathbb{E} \left[ \psi_1 \ b(0,0) \cdot \frac{\Delta W(1)}{\Delta T(1)} \right]$$
A branching Brownian motion

Burger’s equation, nonlinearity:
\( \beta (v v_x - v) \)
Seminlinear equation and branching Brownian motion

- For every $k \in \overline{\mathcal{K}}_T$, introduce

$$\mathcal{W}_k := \mathbb{I}_{\{\theta_k=0\}} + b(T_{k-}, \mathcal{W}_{T_{k-}}) \cdot \frac{\Delta \mathcal{W}_k}{\Delta T_k} \mathbb{I}_{\{\theta_k=1\}}.$$  

Then $u(0, 0) = \mathbb{E}[\psi_n] = \mathbb{E}[\psi]$, where

$$\psi_n := \left[ \prod_{k \in \bigcup_{j=1}^{n} \mathcal{K}^j_T} \frac{g(W^k_T) \mathcal{W}_k}{F(\Delta T_k)} \right] \left[ \prod_{k \in \bigcup_{j=1}^{n} \overline{\mathcal{K}}^j_T \setminus \mathcal{K}^j_T} \frac{a_{l_k}(T_k, W^k_{T_k}) \mathcal{W}_k}{p_{l_k} \rho(T_k)} \right] \left[ \prod_{k \in \overline{\mathcal{K}}_T^{n+1}} (u_{T_{k-}} \mathbb{I}_{\{\theta_k=0\}} + (b \cdot Du)_{T_{k-}} \mathbb{I}_{\{\theta_k=1\}}) \right],$$  

and

$$\psi := \left[ \prod_{k \in \mathcal{K}_T} \frac{g(W^k_T) \mathcal{W}_k}{F(\Delta T_k)} \right] \left[ \prod_{k \in \overline{\mathcal{K}}_T \setminus \mathcal{K}_T} \frac{a_{l_k}(T_k, W^k_{T_k}) \mathcal{W}_k}{p_{l_k} \rho(T_k)} \right].$$
• It only works for small non-linearity $|a_\ell| \ll 1$ or short maturity $T \ll 1$. But approximation of a Lipschitz function $f(u, Du)$ may lead to big non-linearity.

• Locally polynomial approximation:

$$f(u) \approx f_\circ(u, u) = \sum_{\ell=0}^{\ell_0} a_\ell(u) u^\ell.$$ 

• A Picard iteration scheme

$$\partial_t u^{n+1} + \frac{1}{2} \Delta u^{n+1} + f_\circ(u^n, u^{n+1}) = 0.$$
Drawback and extensions

* One can formally deduce an estimator for **fully nonlinear** case, where the nonlinearity is given by

\[
f(u, Du, D^2 u) := \sum_{\ell_0, \ell_1, \ell_2} a_{\ell} u^{\ell_0} (Du)^{\ell_1} (D^2 u)^{\ell_2}.
\]

But the estimator is not integrable because of the Malliavin weight term:

\[
\mathbb{E}[D^2 \varphi(x + W_T)] = \mathbb{E}\left[\varphi(x + W_T) \frac{W_T^2 - T}{T^2}\right].
\]

* See Xavier’s talk ...