Variations on branching methods for non linear PDEs

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Context

Try to solve Semi Linear

\[-\partial_t u - \mathcal{L} u = f(u), u_T = g,\]

Or Full non linear

\[-\partial_t u - \mathcal{L} u = f(u, Du, D^2 u), u_T = g,\]

with

- $f$ polynomial in $u, Du, D^2 u$,
- $g$ Lipschitz, coefficients bounded continuous.

Deterministic methods (Finite Difference...) suffers highly from curse of dimensionality,

Probabilistic methods such to solve BSDE (Semi Linear), SOBSDE (Full linear) can be based on regression, so suffer (to a less extend) from curse of dimensionality.
The semi linear case

Branching dates and particle trajectories (burgers)

Figure: Galton-Watson tree: brownian for Burgers

- \((\tau^k)_{k=(k_1,\ldots,k_n), k_i \in \{1,2\}, n > 1}\) iid rv with density \(\rho\),
- Sequence of branching dates \(T_k=(k_1,\ldots,k_n) = T_k-=(k_1,\ldots,k_{n-1}) + \tau(k_1,\ldots,k_n) \wedge T\),
- \(\Delta T_k = T_k - T_k-\) the time increments
- \((\hat{W}^k)_{k=(k_1,\ldots,k_{n-1},k_n) \in \{1,2\}^n, n > 1}\) independent Brownian motion
- Each particle \(k = (k_1, \ldots, k_{n-1}, k_n) \in \{1, 2\}^n, n > 1\) equipped with a brownian with increments

\[ W^k_{T_k} := W^k_{T_k-} + \hat{W}^k_{\Delta T_k} \]

- Particle position: \(X^k_t := x + \mu t + \sigma_0 W^k_t\) for \(t \in [T_k-, T_k]\)
Original branching methods for semi linear 
(Henry-Labordere et al. [1]) (exemple Burgers) in 1D

- PDE: \( \partial_t u + \frac{1}{2} \sigma_0^2 D^2 u + \mu D u + b u D u = 0 \),
- Using Feyman-Kac:

\[
\begin{align*}
    u(0, x) &= \mathbb{E}_{0,x} \left[ \tilde{F}(T) \frac{g(X_T)}{\tilde{F}(T)} + \int_0^T \frac{b u D u(t, X_t)}{\rho(t)} \rho(t) dt \right] \\
    &= \mathbb{E}_{0,x} \left[ \phi(T(1), X^{(1)}_{T(1)}) \right]
\end{align*}
\]

with \( \tilde{F}(t) := \int_t^\infty \rho(s) ds \) complementary CDF of \( \tau(1) \),

\[
\phi(t, y) := \frac{1_{\{t \geq T\}}}{\tilde{F}(T)} g(y) + \frac{1_{\{t < T\}}}{\rho(t)} (b u D u)(t, y).
\]
Original algorithm for semi linear

- On \( \{ 1_{\{T(1) \geq T\}} \} \) just compute \( \frac{g(X_T)}{F(T)} \).
The semi linear case

Original algorithm for semi linear

On \( \{1 \{T_{(1)} < T\}\} \),

\[
\frac{bu Du(T_{(1)}, X_{T_{(1)}})}{\rho(T_{(1)})} = \frac{b}{\rho(T_{(1)})} EE_{T_{(1)}, X_{T_{(1)}}} [\phi(T_{(1),1}, X^{(1,1)}_{T_{(1),1}})]
\]

\[
D EE_{T_{(1)}, X_{T_{(1)}}} [\phi(T_{(1),2}, X^{(1,p)}_{T_{(1),2}})]
\]

Generate 2 particles \((1, 1)\) marked \(\theta((1, 1)) = 0\) and \((1, 2)\) marked \(\theta((1, 2)) = 1\)
Original algorithm for semi linear

On $\{1\{T(1) < T\}\}$,

$$\frac{buDu(T(1), X_{T(1)})}{\rho(T(1))} = \frac{b}{\rho(T(1))} \mathbb{E}_{T(1), X_{T(1)}} \left[ \phi(T(1,1), X_{T(1,1)}^{(1,1)}) \right]$$

$$D \mathbb{E}_{T(1), X_{T(1)}} \left[ \phi(T(1,2), X_{T(1,2)}^{(1,2)}) \right]$$

Use automatic differentiation:

$$\mathbb{E}_{T(1), X_{T(1)}} \left[ \frac{\hat{W}_{T(1,2)}^{(1,2)}}{\sigma_0 \Delta T_{(1,2)}} \phi(T(1,2), X_{T(1,2)}^{(1,2)}) \right]$$
The semi linear case

Backward recursion for semi linear

\[ \hat{\psi}_k := \frac{g(X^k_T) - g(X^k_{T_k^-})}{F'(\Delta T_k)} \mathbf{1}_{\{\theta_k \neq 0\}} \] if \( T_k = T \),

Control variate

\[ \hat{\psi}_k := \frac{b}{\rho(\Delta T_k)} \prod_{\tilde{k} \in \{(k,1), (k,2)\}} \hat{\psi}_{\tilde{k}} \mathcal{W}_{\tilde{k}}, \] if \( T_k \neq T \)

where

\[ \mathcal{W}_k = \mathbf{1}_{\{\theta_k = 0\}} + \mathbf{1}_{\{\theta_k \neq 0\}} \frac{(\sigma_0)^{-1} \hat{W}^k_{\Delta T_k}}{\Delta T_k}. \]

Weight for \( u \) term \( D_u \) term

\[ u(0, x) = \mathbb{E}_{0, x} \left[ \hat{\psi}(1) \right]. \]
The semi linear case

Finite variance requirement

\[ \frac{1}{x \rho(x)^2} = O(x^\alpha) \text{ as } x \to 0 \text{ with } \alpha \geq 0. \text{ Use Gamma laws parameters } \kappa \leq 0.5, \]

Implies a lot of branching and a higher computational cost.

only finite for small coefficients and small maturities.

Modification to handle longer maturities:

- Use exponential law for \( u \) terms, use gamma law for \( Du \) term.

- Compute conditional expectation with 2 or 4 particles.
The semi linear case

Results on burgers type equation dimension 4

Figure: Estimation and error in $d = 4$. Maturity $T = 1.5$

Figure: Error in $d = 4$, maturity $T = 2$, $T = 2.5$
For gradient term:

\[
\mathbb{E}_{T(1), X_{T(1)}} \left[ \frac{\hat{W}^{(1,p)}_{\Delta T^{(1,p)}}}{\sigma_0 \Delta T^{(1,p)}} \left( \phi(T^{(1,p)}, X^{(1,p)}_{T(1,p)}) - \phi(T^{(1,p)}, X^{(1,p^1)}_{T(1,p)}) \right) \right],
\]

\[p = 1, 2\]

\[X^{(1,p^1)}\] has the same past as \[X^{(1,p)}\] at date \[T(1)\],

same increment between \[T^{(1,p)}\] and \[T\],

no brownian increment between \[T(1)\] and \[T^{(1,p)}\]

Acts as a control variate.

\[
\mathbb{E}_{T(1), X_{T(1)}} \left[ \left( \phi(T^{(1,p)}, X^{(1,p)}_{T(1,p)}) - \phi(T^{(1,p)}, X^{(1,p^1)}_{T(1,p)}) \right)^2 \right] = O(\Delta T^{(1,p)}).
\]

Permits to use all \(\rho\) densities (so exponential); finite variance in the linear case. No current result in the semi linear one.

This ghost method outperforms the original method.
The semi linear case

Original Galton-Watson tree and the ghost particles associated

(a) Original Galton-Watson tree

\[
\begin{align*}
W^{(1)} &= \hat{W}^{(1)} \\
W^{(1,1)} &= \hat{W}^{(1)} + \hat{W}^{(1,1)} \\
W^{(1,2)} &= \hat{W}^{(1)} + \hat{W}^{(1,2)} \\
W^{(1,1,1)} &= \hat{W}^{(1)} + \hat{W}^{(1,1)} + \hat{W}^{(1,1,1)} \\
W^{(1,1,2)} &= \hat{W}^{(1)} + \hat{W}^{(1,1)} + \hat{W}^{(1,1,2)}
\end{align*}
\]

(b) Tree with ghost particle

\[
\begin{align*}
k &= (1,1^1) \\
W^{(1)} &= \hat{W}^{(1)} \\
W^{(1,1)} &= \hat{W}^{(1)} \\
W^{(1,2)} &= \hat{W}^{(1)} + \hat{W}^{(1,2)} \\
W^{(1,1,1)} &= \hat{W}^{(1)} + \hat{W}^{(1,1)} + \hat{W}^{(1,1,1)} \\
W^{(1,1,2)} &= \hat{W}^{(1)} + \hat{W}^{(1,1,2)}
\end{align*}
\]
The semi linear case

Original re-normalization for burgers Labordère et al. [2]

\[ \hat{\psi}_k := \frac{g(X_T^k)}{F(\Delta T_k)} \text{ if } T_k = T \]

\[ \hat{\psi}_k := \frac{b}{\rho(\Delta T_k)} \prod_{\tilde{k} = \{(k,1),(k,2)\}} (\hat{\psi}_{\tilde{k}} - \hat{\psi}_{\tilde{k}1} 1_{\{\theta(\tilde{k}) \neq 0\}}) \mathcal{W}_{\tilde{k}}, \text{ if } T_k < T \]

\[ u(0, x) = \mathbb{E}_{0,x} \left[ \hat{\psi}_{(1)} \right]. \]
The semi linear case

Re-normalization with antithetic ghosts Warin [3]

\[ E_{T(1), X_{T(1)}} \left[ \left( \sigma_0^\top \right)^{-1} \frac{\hat{W}^{(1,p)}_{\Delta T(1,p)}}{\Delta T(1,p)} \frac{1}{2} \left( \phi(T(1,p), X^{(1,p)}_{T(1,p)}) - \phi(T(1,p), X^{(1,p^1)}_{T(1,p)}) \right) \right]. \]

- \( X^{(1,p^1)} \) has the same past as \( X^{(1,p)} \) at date \( T(1) \),

same increment between \( T(1,p) \) and \( T \) and

\[ -\frac{\hat{W}^{(1,p)}_{\Delta T_k}}{\Delta T_k} \] increment between \( T(1) \) and \( T(1,p) \).

- Finite variance in the linear case.
Numerical original ghost versus antithetic ghosts for $u$ calculation.

**Figure:** Error in $d = 6$ $T = 3$ for Burgers

**Figure:** Error in $d = 6$ for $(Du)^2$ non linearity.
Numerical original ghost versus antithetic ghosts for $D_{\mu}$ calculation.

**Figure:** Error in $d = 6$ for the term $b_{D_{\mu}}$ on Burgers test case for $T = 1.5$. 
The full non linear case

Full non linear \( f(u, Du, D^2u) = bu^l_0 (Du)^l_1 (D^2u)^l_2 \):

original scheme with 2 ghosts Labordère et al. [2]

\[
D^2 \mathbb{E}_{T(1)},X_{T(1)} [\phi(T(1,p), X_{T(1,p)}^{(1,p)})] = \mathbb{E}_{T(1)},X_{T(1)} [(\sigma_0)^{-2} \frac{(\hat{W}^{(1,p)}_{\Delta T(1,p)})^2 - \Delta T(1,p)}{\Delta T(1,p)^2} \psi],
\]

\[
\psi = \frac{1}{2} [\phi(T(1,p), X_{T(1,p)}^{(1,p)}) + \phi(T(1,p), X_{T(1,p)}^{(1,p)}) - 2\phi(T(1,p), X_{T(1,p)}^{(1,p)})].
\]

- \( X_{T(1,p)}^{(1,p)} \) the original particle
- \( X_{T(1,p)}^{(1,p^1)} \) ghost with \( -\hat{W}^{(1,p)}_{\Delta T_k} \) increment between \( T(1) \) and \( T(1,p) \)
- \( X_{T(1,p)}^{(1,p^2)} \) ghost without increment between \( T(1) \) and \( T(1,p) \)
The variance of the scheme is finite for small maturities, small coefficients,

No current proof for the full non linear case.
A first new scheme for Full Non Linear with 3 ghosts

Warin [3]

Use first order derivative weights on two successive time steps $\Delta T_{(1,p)} \frac{\Delta T_{(1,p)}}{2}$.

$$(\hat{W}^k)_{k=(k_1,\cdots,k_{n-1},k_n)\in \mathbb{N}^n, n>1, i=1,2} \text{ independent BM}$$

$$D^2\mathbb{E}_{T(1),X_T(1)} \left[ \phi(T_{(1,p)},X_{T_{(1,p)}}) \right] = \mathbb{E}_{T(1),X_T(1)} \left[ 2(\sigma_0)^{-2} \frac{\hat{W}_{(1,p)^1}}{\Delta T_{(1,p)}} \frac{\hat{W}_{(1,p)^2}}{\Delta T_{(1,p)}} \psi \right]$$

$$\psi = \phi(T_{(1,p)},X_{T_{(1,p)}}) + \phi(T_{(1,p)},X_{T_{(1,p)}}^{(1,p^3)}) - \phi(T_{(1,p)},X_{T_{(1,p)}}^{(1,p^1)}) - \phi(T_{(1,p)},X_{T_{(1,p)}}^{(1,p^2)})$$

$$X^{(1,p)} = X^{(1)} + \mu \Delta T_{(1,p)} + \sigma_0 \left[ \frac{\hat{W}_{(1,p)^1}}{\Delta T_{(1,p)}} + \frac{\hat{W}_{(1,p)^2}}{\Delta T_{(1,p)}} \right]$$

$$X^{(1,p^3)} = X^{(1)} + \mu \Delta T_{(1,p)} \text{ ghost freezing position}$$

$$X^{(1,p^1)} = X^{(1)} + \mu \Delta T_{(1,p)} + \sigma_0 \left[ \frac{\hat{W}_{(1,p)^1}}{\sqrt{2}} \right] \text{ ghost without second } \hat{W} \text{ increment}$$

$$X^{(1,p^2)} = X^{(1)} + \mu \Delta T_{(1,p)} + \sigma_0 \left[ \frac{\hat{W}_{(1,p)^2}}{\sqrt{2}} \right] \text{ ghost without first } \hat{W} \text{ increment}$$
Remark and extension

- Bounds on variance calculation indicate a potential smaller variance value of the new scheme,
- An antithetic ghost version of the second scheme with 7 ghosts can be used.
- Higher number of ghosts means higher memory requirement.
- Higher derivatives are easy to treat.
The full non linear case

Results for full non linearity $u D^2 u$

\[
F(u, Du, D^2 u) = h(t, x) + \frac{0.1}{d} u(1 : D^2 u),
\]

\[
\mu = 0.21 \sigma_0 = 0.51, \quad \alpha = 0.2
\]

\[
h(t, x) = (\alpha + \frac{\sigma_0^2}{2}) \cos(x_1 + \ldots + x_d) e^{\alpha(T-t)} + 0.1 \cos(x_1 + \ldots + x_d)^2 e^{2\alpha(T-t)} + \mu \sin(x_1 + \ldots + x_d) e^{\alpha(T-t)},
\]

\[
u(t, x) = \cos(x_1 + \ldots + x_d) e^{\alpha(T-t)}.
\]

Figure: Solution $u(0, 0.5)$ obtained and error in $d = 6$ with $T = 1$, analytical solution is $-1.20918$. 
Results for full non linearity $uD^2u$: derivative

Figure: Derivative $(1.Du)$ obtained and error in $d = 6$ with $T = 1$. 
The full non linear case

Results for non linearity $DuD^2u$

\[ f(u, Du, D^2u) = 0.0125(1.Du)(1 : D^2u). \]

**Figure:** Solution $u(0, 0.5)$ and error obtained for $d = 4$ with $T = 1$. 

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