When Almgren Chris meet Black Scholes
derivative hedging under illiquid market

Jiang Pu

Joint work with O. Guéant. Work supported by HSBC France within the Research Initiative “Modélisation des marchés financiers à haute fréquence” (formerly “Exécution optimale et statistiques de la liquidité haute fréquence”)

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Outline

1. Model and Notations
   - Introduction
   - Notations

2. Optimization Problem
   - Payoff
   - Optimization Problem
   - Change of Variables

3. Numerical Experiments
   - Reference Scenario
   - Importance of Initial Position
   - Effect of Permanent Impact
   - Comparative Statics
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**Introduction**

Classical framework: Black-Scholes or Bachelier → frictionless market, price-taker agent

Continuous trading

Pricing using a risk-neutral expectation

Replicating portfolio

Replicating strategy given by the $\Delta$ of the option

Liquidity-related questions:
- options written on illiquid assets?
- large nominal?
- large $\Gamma$?
- difference between physical and cash settlement?
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**Classical framework: Black-Scholes or Bachelier**
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Framework

Model:
- Continuous time modelling
- Almgren-Chriss-like market impacts
- Expected utility framework (CARA)
- Indifference pricing
- Partial differential equation (viscosity solutions of HJB)

Features:
- Market impact and execution costs
- Partial hedge
- Modelling Physical delivery
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When Almgren Chris meet Black Scholes derivative hedging under
We consider that we have sold this call option with physical settlement.

Characteristics of the Call option

- Strike $K$
- Maturity $T$
- Nominal $N$ (in shares)
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**Characteristics of the Call option**
- Strike $K$
- Maturity $T$
- Nominal $N$ (in shares)

*N matters because the introduction of execution costs and market impact makes the problem a non-linear one.*
Notations

- Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a filtered probability space.
- The number of shares in the hedging portfolio
  \[
  q_t = q_0 + \int_0^t v_s ds,
  \]
  where \(q_0\) denotes the number of shares in the portfolio at inception.
- The process \((v)\) lies in the admissible set:
  \[
  \mathcal{A} = \left\{ v \in \mathcal{P}(0, T), \int_0^T |v_t| dt \in L^\infty(\Omega) \right\}
  \]
  where \(\mathcal{P}(s, t)\) is the set of \(\mathbb{R}\)-valued progressively measurable processes on \([s, t]\).
We could consider the classical linear model:

**Linear Permanent Market Impact Model**

\[
dS_t = \sigma dW_t + kv_t dt
\]

Here we consider the more general framework:

**General Permanent Market Impact Model**

\[
dS_t = \sigma dW_t + f (|q_0 - q_t|) v_t dt
\]

where \( f : \mathbb{R}_+^* \to \mathbb{R}_+ \) nonincreasing and in \( L^1_{loc}(\mathbb{R}_+) \).
Instantaneous Market Impact (Execution costs)

Execution Costs Function

Execution costs are modeled by $L \in C(\mathbb{R}, \mathbb{R}_+)$ verifying:
- $L(0) = 0$
- $L$ is an even function
- $L$ is increasing on $\mathbb{R}_+$
- $L$ is strictly convex
- $L$ is asymptotically superlinear: $\lim_{\rho \to +\infty} \frac{L(\rho)}{\rho} = +\infty$

In practice, $L$ is a function of the following form:

$$L(\rho) = \eta |\rho|^{1+\phi} + \psi |\rho|^{1+\psi}$$
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In practice

$L$ is a function of the following form:

$$L(\rho) = \eta|\rho|^{1+\phi} + \psi|\rho|$$
Impact on Cash

The Cash account $X$ evolves as:

$$dX_t = -v_t S_t dt - V_t L \left( \frac{v_t}{V_t} \right) dt,$$

where $X_0 = 0$

where the process $(V_t)_t$ is the market volume process, assumed to be deterministic, positive and bounded.
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Payoff

We assume physical settlement. The payoff is then given by the following rule:

If the option is exercised:

\[ X_T - (N - q_T) S_T + L(q_T, N) \]

where \( L(q_T, q_T') \) models costs at time \( T \) to go from a portfolio with \( q \) shares to a portfolio with \( q' \) shares.

Otherwise, the payoff is:

\[ X_T + q_T S_T - L(q_T, 0) \]

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When Almgren Chris meet Black Scholes derivative hedging under
We assume physical settlement. The payoff is then given by the following rule:

- If the option is exercised:

\[
X_T + KN - ((N - q_T)S_T + \mathcal{L}(q_T, N))
\]

where \(\mathcal{L}(q, q')\) models costs at time \(T\) to go from a portfolio with \(q\) shares to a portfolio with \(q'\) shares.
We assume physical settlement. The payoff is then given by the following rule:

- If the option is exercised:
  \[
  X_T + KN - ((N - q_T)S_T + \mathcal{L}(q_T, N))
  \]
  where \(\mathcal{L}(q, q')\) models costs at time \(T\) to go from a portfolio with \(q\) shares to a portfolio with \(q'\) shares.

- Otherwise, the payoff is:
  \[
  X_T + q_T S_T - \mathcal{L}(q_T, 0)
  \]
  gain of selling the \(q_T\) shares
The payoff can then be written as (the threshold is assumed to be $K' \leq K$):

$$X_T + q_T S_T + 1_{S_T \geq K'} (N(K - S_T) - \mathcal{L}(q_T, N)) - 1_{S_T < K'} \mathcal{L}(q_T, 0)$$
The payoff can then be written as (the threshold is assumed to be $K' \leq K$):

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To ensure the absence of dynamic arbitrage, we specify $\mathcal{L}$ as:

$$\mathcal{L}(q, q') = \ell(q' - q) + q'(G(q') - G(q)) - (F(q') - F(q))$$

where $\ell$ is the cost function when there is no permanent market impact, $F(q) = \int_{q_0}^{q} zf(|q_0 - z|)dz$ and $G(q) = \int_{q_0}^{q} f(|q_0 - z|)dz$. 
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The bank maximizes its expected utility:
\[
\sup_{\nu \in \mathcal{A}} \mathbb{E} \left[ - \exp \left( - \gamma \left( X_T + q_T S_T + 1_{S_T \geq K'} (N(K - S_T) - \mathcal{L}(q_T, N)) - 1_{S_T < K'} \mathcal{L}(q_T, 0) \right) \right) \right]
\]
where \( \gamma \) is the absolute risk aversion parameter of the bank.

This framework permits to define a price for the option using indifference pricing.
To solve this problem, we introduce the value function \( u \):

\[
\begin{align*}
\text{Definition} \\

u(t, x, q, S) &= \sup_{\nu \in \mathcal{A}_t} \mathbb{E} \left[- \exp \left(-\gamma \left( X_{T}^{t, x, \nu} + q_{T}^{t, q, \nu} S_{T}^{t, S, \nu} \right) \right.ight. \\
&\quad \left. + 1_{S_{T}^{t, S, \nu} \geq K'} \left(N(K - S_{T}^{t, S, \nu}) - \mathcal{L}(q_{T}^{t, q, \nu}, N)\right) \right] \\
&\quad - 1_{S_{T}^{t, S, \nu} < K'} \mathcal{L}(q_{T}^{t, q, \nu}, 0) \right],
\end{align*}
\]

where \( \mathcal{A}_t = \left\{ \nu \in \mathcal{P}(t, T), \int_t^T |\nu_s| ds \in L^\infty(\Omega) \right\} \).
HJB Equation

$u$ satisfies the following HJB equation:

$$-\partial_t u - \frac{1}{2} \sigma^2 \partial^2_{SS} u - \sup_{v \in \mathbb{R}} \left\{ v \partial_q u + \left( -vS - L \left( \frac{v}{V_t} \right) V_t \right) \partial_x u \right. $$

$$ \left. -\partial_S uf(|q_0 - q|)v \right\} = 0$$

with terminal condition:

$$u(T, x, q, S) = -\exp \left( -\gamma \left( x + qS - 1_{S < K'} L(q, 0) \right) ight.$$ 

$$+ 1_{S \geq K'} (N(K - S) - L(q, N)) \right)$$
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Change of Variables

We use the following change of variables:

\[
\theta(t, q, S) = -\exp(-\gamma(x + qS - F(q) - \theta(t, q, S - G(q))))
\]
We use the following change of variables:

**Definition**

We introduce $\theta$ by:

$$u(t, x, q, S) = - \exp(-\gamma(x + qS - F(q) - \theta(t, q, S - G(q))))$$

**Indifference Price**

$\theta(0, q_0, S_0)$ is the indifference price to write the call option.
The PDE satisfied by $\theta$ is the following:

$$
-\partial_t \theta - \frac{1}{2} \sigma^2 \partial^2_{\tilde{S}\tilde{S}} \theta - \frac{1}{2} \gamma \sigma^2 (\partial_{\tilde{S}} \theta - q)^2 + V_t H(\partial_q \theta) = 0
$$

where $H$ is the Legendre transform of $L$:

$$
H(p) = \sup_{\rho} \{p \rho - L(\rho)\}
$$

Terminal condition:

$$
\theta(T, q, \tilde{S}) = 1_{\tilde{S} \geq K'} \left( N(\tilde{S} - K) + NG(N) - F(N) \right)
$$

$$
+ \ell(N - q) \right) + 1_{\tilde{S} < K'} \left( \ell(q) - F(0) \right)
$$
Interpretation of the PDE:

\[ -\partial_t \theta - \frac{1}{2} \sigma^2 \partial^2_{\tilde{S} \tilde{S}} \theta - \frac{1}{2} \gamma \sigma^2 (\partial_{\tilde{S}} \theta - q)^2 + V_t H(\partial_q \theta) = 0 \]

An optimal control is given by:

\[ v^*(t, q, S) = H'(\partial_q \theta(t, q, S - G(q))) \]
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Reference Scenario

- $S_0 = K = K' = 45\, \text{\euro}$
- $\sigma = 0.6\, \text{\euro}\cdot\text{day}^{-1/2}$ (\approx 21\% annual volatility)
- $T = 60\, \text{days}$
- $V = 1\,000\,000\,\text{shares} \cdot \text{day}^{-1}$
- $V_d = 1\,000\,000\,\text{shares}$
- $N = 5\,000\,000\,\text{shares}$
- $L(\rho) = \eta|\rho|^{1+\phi}$ with $\eta = 0.1\, \text{\euro} \cdot \text{shares}^{-1} \cdot \text{day}^{-1}$ and $\phi = 0.75$
- $\gamma = 10^{-6}\,\text{\euro}^{-1}$
Reference Scenario 1 (No PMI)

**Figure:** Reference Scenario 1 - Stock Price and Optimal Strategy
Reference Scenario 2 (No PMI)

Figure: Reference Scenario 2 - Stock Price and Optimal Strategy
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Importance of Initial Position

Figure: Optimal portfolio when prices follow trajectory 1
Importance of Initial Position

Figure: Optimal portfolio when prices follow trajectory 2
The prices are given by:

<table>
<thead>
<tr>
<th></th>
<th>( q_0 = 0 )</th>
<th>( \frac{q_0}{N} = 0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price of the call</td>
<td>2.54</td>
<td>2.19</td>
</tr>
<tr>
<td>Implied ( \sigma ) in the Bachelier model</td>
<td>0.82</td>
<td>0.71</td>
</tr>
</tbody>
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Building the initial position in \( \Delta \) is usually costly.
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When Almgren Chris meet Black Scholes derivative hedging under
Effect of Permanent Impact

We add the following permanent impact:

Permanent Impact

\[ f(q) = 0.001 \frac{1}{\sqrt{|q|}} \]
Trajectory 1 - Strategies

Figure: Optimal portfolio when prices follow trajectory 1
Trajectory 1 - Impacted Prices

Figure: Prices (trajectory 1) with the influence of permanent market impact
Figure: Optimal portfolio when prices follow trajectory 2
**Figure:** Prices (trajectory 2) with the influence of permanent market impact.
Effect of Permanent Impact

- Mechanical effect: if the price goes up (resp. down), our position goes up (resp. down), and it pushes the price up (resp. down).

- Strategical effect: near maturity and near the money, the bank has incentives to sell shares to push down the price so that the option expires worthless.
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Execution Costs

Figure: Optimal portfolio for different values of $\eta$, when prices follow Trajectory 1
Execution Costs

Figure: Optimal portfolio for different values of $\eta$, when prices follow Trajectory 2
Execution Costs

When $\eta$ increases:

- The trajectories are smoother
- They are closer to the position $0.5N$ to avoid round trips

When $\eta \to 0$, we obtain the limiting case of $\Delta$-Hedging.

The prices are given by:

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>0.2</th>
<th>0.1</th>
<th>0.05</th>
<th>0.01</th>
<th>0 (Bachelier)</th>
</tr>
</thead>
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<tr>
<td>Price of the call</td>
<td>2.36</td>
<td>2.19</td>
<td>2.07</td>
<td>1.92</td>
<td>1.85</td>
</tr>
<tr>
<td>Implied $\sigma$ in a Bachelier model</td>
<td>0.76</td>
<td>0.71</td>
<td>0.67</td>
<td>0.62</td>
<td>0.6</td>
</tr>
</tbody>
</table>

- Prices are higher when $\eta$ increases.
**Bid-Ask Spread**

**Figure:** Optimal portfolio in presence of bid-ask spread, when prices follow Trajectory 1
Figure: Optimal portfolio in presence of bid-ask spread, when prices follow Trajectory 1
Risk Aversion

Figure: Optimal portfolio for different values of $\gamma$, when prices follow trajectory 1

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Risk Aversion

Figure: Optimal portfolio for different values of $\gamma$, when prices follow trajectory 2.
Questions?