Game options in an imperfect financial market with default and model uncertainty

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Game options

Definition

- Derivative contracts, introduced by Kifer in 2000, which can be terminated by both counterparties at any time before maturity $T$.
- Extend the setup of American options by allowing the seller to cancel the contract.

If the buyer exercises at time $\tau$, he gets $\xi_\tau$ from the seller,
but if the seller cancels at $\sigma$ before $\tau$, he pays $\zeta_\sigma \geq \xi_\sigma$ to the buyer.

In short, if the buyer exercises at a stopping time $\tau \leq T$ and the seller cancels at a stopping time $\sigma \leq T$, then the seller pays to the buyer the payoff $\xi_\tau 1_{\tau \leq \sigma} + \zeta_\sigma 1_{\tau > \sigma}$ at terminal time $\tau \wedge \sigma$.

The difference $\zeta_t - \xi_t$ for all $t$ and is interpreted as a penalty for the seller for cancellation of the contract.
In the case of perfect markets, Kifer introduces the *fair price* $u_0$ of the game option, as the minimum initial wealth for the seller to cover his liability to pay the payoff to the buyer until a cancellation time, whatever the buyer’s exercise time.

He shows both in the CRR discrete-time and Black-Scholes model (with $\xi$ and $\zeta$ continuous), that $u_0$ is equal to the value function of a Dynkin game:

$$u_0 = \sup_{\tau} \inf_{\sigma} \mathbb{E}_Q[\tilde{\xi}_\tau \mathbf{1}_{\tau \leq \sigma} + \tilde{\zeta}_\sigma \mathbf{1}_{\tau > \sigma}] = \inf_{\sigma} \sup_{\tau} \mathbb{E}_Q[\tilde{\xi}_\tau \mathbf{1}_{\tau \leq \sigma} + \tilde{\zeta}_\sigma \mathbf{1}_{\tau > \sigma}],$$

where $\tilde{\xi}_t$ and $\tilde{\zeta}_t$ are the discounted values of $\xi_t$ and $\zeta_t$, and $Q$ is the unique martingale probability measure.
Our goal

1. Study game options (pricing and superhedging) in the case of imperfections in the market taken into account via the nonlinearity of the wealth dynamics. We moreover include the possibility of a default and irregular payoffs (RCLL only).

2. Study game options under model uncertainty, in particular ambiguity on the default probability.
Financial market with default

Let \((\Omega, \mathcal{G}, \mathbb{P})\) be a complete probability space. Consider a market with three assets with price process \(S = (S^0, S^1, S^2)\):

\[
\begin{aligned}
    dS^0_t &= S^0_t r_t dt \\
    dS^1_t &= S^1_t [\mu^1_t dt + \sigma^1_t dW_t] \\
    dS^2_t &= S^2_t [\mu^2_t dt + \sigma^2_t dW_t - dM_t],
\end{aligned}
\]

- \(W\) is a unidimensional standard Brownian motion
- \(M_t = N_t - \int_0^t \lambda_s ds\) is the compensated martingale of the jump process \(N_t := 1_{\vartheta \leq t}, \ t \in [0, T]\), where \(\vartheta\) is a r. v. modeling a default time. This default can appear at any time, i.e. \(\mathbb{P}(\vartheta \geq t) > 0 \ \forall t \in [0, T]\).

Let \(\mathcal{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}\) the augmented filtration generated by \(W\) and \(N\). Suppose \(W\) is a \(\mathcal{G}\)-Brownian motion. 

*Process \(S^2\) is the price of a defaultable asset with total default. Vanishes after \(\vartheta\).*

\(\sigma^1, \sigma^2, r, \mu^1, \mu^2\) predictable; \(\sigma^1, \sigma^2 > 0; r, \sigma^1, \sigma^2, \mu^1, \mu^2, \lambda, \lambda^{-1}, (\sigma^1)^{-1}, (\sigma^2)^{-1}\) bounded.
Option pricing in the perfect market case

Consider an investor with initial wealth $x$ and risky asset strategy $\varphi = (\varphi^1, \varphi^2)$. Let $V^x,\varphi_t$ the value of the portfolio at time $t$.

**Self financing condition:**

$$dV_t = (r_t V_t + \varphi_t^1 \sigma_t^1 \theta_t^1 - \varphi^2_t \theta^2_t \lambda_t) dt + \varphi'_t \sigma_t dW_t - \varphi^2_t dM_t,$$

where $\theta_t^1 := (\mu_t^1 - r_t)(\sigma_t^1)^{-1}$; $\theta_t^2 := -(\mu_t^2 - \sigma_t^2 \theta_t^1 - r_t)\lambda_t^{-1} 1_{\{t\leq \vartheta\}}$.

Consider a **European option** with maturity $T$ and payoff $\xi$ in $L^2(G_T)$.

The unique solution $(X, Z, K) \in S^2 \times H^2 \times H^2_\lambda$ of the $\lambda$-linear BSDE (DQS'16)

$$-dX_t = -(r_t X_t + (Z_t + \sigma_t^2 K_t) \theta_t^1 + K_t \theta^2_t \lambda_t) dt - Z_t dW_t - K_t dM_t; \ X_T = \xi.$$

provides the **replicating** portfolio: $\varphi'_t \sigma_t = Z_t$; $-\varphi^2_t = K_t$.

This defines a change of variables:

$$\Phi(Z, K) := \varphi = (\varphi^1, \varphi^2) \text{ with } \varphi^2_t = -K_t; \varphi^1_t = (Z_t + \sigma_t^2 K_t)(\sigma_t^1)^{-1}.$$
$X = X(\xi)$ coincides with $V^{X_0,\varphi}$, the value of the (hedging) portfolio associated with initial wealth $x = X_0$ and portfolio strategy $\varphi$. We have:

$$X_t(\xi) = \mathbb{E}[e^{-\int_t^T r_s ds} \zeta_{t,T} | \mathcal{G}_t],$$

where $\zeta$ satisfies

$$d\zeta_{t,s} = \zeta_{t,s} \left[ -\theta^1_s dW_s - \theta^2_s dM_s \right]; \quad \zeta_{t,t} = 1,$$

with $\theta^1_t := \left( \mu^1_t - r_t \right) \left( \sigma^1_t \right)^{-1}$; $\theta^2_t := -\left( \mu^2_t - \sigma^2_t \theta^1_t - r_t \right) \lambda_t^{-1} 1_{\{t \leq \vartheta\}}$.

This defines a linear price system $X: \xi \mapsto X(\xi)$.

When $\theta^2_t < 1$, $0 \leq t \leq \vartheta$, $dt \otimes dP$-a.s. Then $\zeta_{t,\cdot} > 0$.

The probability $Q$ which admits $\zeta_{0,T}$ as density on $\mathcal{G}_T$, is the unique martingale probability measure.

In this case, the price system $X$ is increasing and corresponds to the classical free-arbitrage price system.
The imperfect market model $\mathcal{M}^g$

Consider now the case of imperfections in the market taken into account via the nonlinearity of the dynamics of the wealth $V_t^{x,\varphi}$:

$$-dV_t = g(t, V_t, \varphi_t'\sigma_t, -\varphi_t^2)dt - \varphi_t'\sigma_t dW_t + \varphi_t^2 dM_t, \quad V_0 = x$$

or equivalently, setting $Z_t = \varphi_t'\sigma_t$ and $K_t = -\varphi_t^2$,

$$-dV_t = g(t, V_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t.$$ 

In a perfect market, $g(t, x, z, k) = -r_t x - (z + \sigma^2_t k)\theta^1_t - \theta^2_t \lambda_t k$.

Here $g$ is a nonlinear $\lambda$-admissible driver, i.e. measurable, $g(., 0, 0, 0) \in \mathbb{H}^2$,

$$|g(\omega, t, y_1, z_1, k_1) - g(\omega, t, y_2, z_2, k_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + \sqrt{\lambda_t}|k_1 - k_2|).$$
Examples of market imperfections

- **Different borrowing and lending interest rates** $R_t$ and $r_t$ with $R_t \geq r_t$

  
  \[ g(t, V_t, \varphi_t \sigma_t, -\varphi_t^2) := -(r_t V_t + \varphi_t^1 \theta_t^1 \sigma_t^1 - \varphi_t^2 \lambda_t \theta_t^2) + (R_t - r_t)(V_t - \varphi_t^1 - \varphi_t^2)^-, \]

  where $\varphi_t^2$ vanishes after $\varphi_t$.

- **Large investor seller** whose trading strategy $\varphi_t$ affects the market

  \[ g(t, V_t, \varphi_t \sigma_t, -\varphi_t^2) := - \bar{r}(t, V_t, \varphi_t) V_t - \varphi_t^1 (\bar{\theta}^1 \bar{\sigma}^1)(t, V_t, \varphi_t) + \varphi_t^2 \lambda_t \bar{\theta}^2(t, V_t, \varphi_t). \]

- **Taxes on risky investments profits**

  \[ g(t, V_t, \varphi_t \sigma_t, -\varphi_t^2) := - (r_t V_t + \varphi_t^1 \theta_t^1 \sigma_t^1 - \varphi_t^2 \theta_t^2 \lambda_t) + \rho (\varphi_t^1 + \varphi_t^2)^+. \]

  Here, $\rho \in ]0, 1[$ represents an instantaneous tax coefficient.
Nonlinear pricing in the imperfect market $\mathcal{M}^g$

Consider a European option with maturity $T$ and terminal payoff $\xi \in L^2(G_T)$. $\exists! (X, Z, K)$ in $S^2 \times H^2 \times H^2$ solution of the BSDE

$$-dX_t = g(t, X_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t; \quad X_T = \xi.$$

The process $X$ is equal to the wealth process associated with initial value $x = X_0$ and strategy $\varphi = \Phi(Z, K)$, that is $X = V^{X_0, \varphi}$.

Its initial value $X_0 = X_0(T, \xi)$ is thus a sensible price at time 0 for the seller of the claim $\xi$ since it allows him to construct a hedging strategy $\varphi$ s.t. the value of the associated portfolio is equal to $\xi$ at time $T$. Similarly for $X_t = X_t(T, \xi)$ and it is the unique price which satisfying the hedging property.

This leads to a nonlinear pricing system, first introduced in NEK-Quenez'96 in Brownian framework, later called $g$-evaluation and denoted by $\mathcal{E}^g$:

$$\forall S \in [0, T], \forall \xi \in L^2(G_S)$$

$$\mathcal{E}_{t,S}^g(\xi) := X_t(S, \xi), t \in [0, S].$$
To ensure (strict) monotonicity and the no arbitrage property of the nonlinear pricing system $\mathcal{E}^g$, we assume

$$g(t, x, z, k_1) - g(t, x, z, k_2) \geq \gamma_{t, x, z, k_1, k_2}(k_1 - k_2)\lambda_t,$$

with $\gamma_{t, x, z, k_1, k_2} > -1$.

This is satisfied e.g. if $g$ is non-decreasing wrt $k$, or if $g$ is $C^1$ in $k$ with $\partial_k g(t, \cdot) > -\lambda_t$ on $\{t \leq \vartheta\}$.

In the special case of perfect market, $\partial_k g(t, \cdot) = -\theta_t^2$, and this Assumption is equivalent to $\theta_t^2 < 1$, the usual assumption.
Game options in the imperfect market $\mathcal{M}^g$

**Definition**

The game option consists for the seller to select a cancellation time $\sigma \in \mathcal{T}$ and for the buyer an exercise time $\tau \in \mathcal{T}$, so that the seller pays to the buyer at time $\tau \land \sigma$ the payoff

$$I(\tau, \sigma) := \xi_{\tau} 1_{\tau \leq \sigma} + \zeta_{\sigma} 1_{\sigma < \tau}.$$ 

**Assumptions:** $\xi, \zeta$ : adapted RCLL processes in $S^2$ with $\zeta_T = \xi_T$; $\xi_t \leq \zeta_t, 0 \leq t \leq T$ a.s. satisfying **Mokobodzki’s condition:**

$\exists$ two nonnegative RCLL supermartingales $H$ and $H'$ in $S^2$ such that:

$$\xi_t \leq H_t - H'_t \leq \zeta_t \quad 0 \leq t \leq T \quad \text{a.s.}$$

(holds e.g. when $\xi$ or $\zeta$ is a semimartingale satisfying some integrability conditions).
Game options in the imperfect market $M^g$

**Definition 1:** For each initial wealth $x$, a **super-hedge** against the game option is a pair of stopping time $\sigma$ and portfolio strategy $\varphi \in H^2 \times H^2_\lambda$ s.t.

\[
V^x_t,\varphi \geq \xi_t, \quad 0 \leq t \leq \sigma \quad \text{a.s. and} \quad V^x_{\sigma},\varphi \geq \zeta_\sigma \quad \text{a.s.}
\]

**Definition 2:** Define the **seller's price** of the game option as

\[
u_0 := \inf \{ x \in \mathbb{R}, \; \exists (\sigma, \varphi) \in S(x) \},
\]

where $S(x)$ is the set of all super-hedges associated with initial wealth $x$. If inf is attained, $u_0$ is called the super-hedging price.

**Definition 3:** We define the **$g$-value** of the game option as

\[
\inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau \wedge \sigma}[I(\tau, \sigma)].
\]
Game options in the imperfect market $\mathcal{M}^g$

Main results (1/2)

The seller’s price $u_0$ of the game option is equal to the $g$-value.

Steps:

1. The $g$-value is equal to the value of the generalized Dynkin game

$$\inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^{g}_{0,\tau \land \sigma} [l(\tau, \sigma)] = \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathcal{E}^{g}_{0,\tau \land \sigma} [l(\tau, \sigma)].$$

2. This value is equal to $Y_0$, where $(Y, Z, K, A, A')$ is the unique solution in $\mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\lambda \times A^2 \times A^2$ of the Doubly Reflected BSDE associated with barriers $\xi$ and $\zeta$ and driver $g$.

3. $u_0 = Y_0$. 
Game options in the imperfect market $\mathcal{M}^g$

Associated Doubly Reflected BSDE

\[-dY_t = g(t, Y_t, Z_t, K_t)\,dt + dA_t - dA'_t - Z_t\,dW_t - K_t\,dM_t; \ Y_T = \xi_T,
\]

(i) $\xi_t \leq Y_t \leq \zeta_t$, $0 \leq t \leq T$ a.s.,

(ii) $dA_t \perp dA'_t$ \ (the measures are mutually singular)

(iii) $\int_0^T (Y_t - \xi_t)\,dA^c_t = 0$ a.s. and $\int_0^T (\zeta_t - Y_t)\,dA'^c_t = 0$ a.s.

$\Delta A^d_T = \Delta A^d_T 1_{\{Y_T = \xi_T\}}$ and $\Delta A'^d_T = \Delta A'^d_T 1_{\{Y_T = \zeta_T\}}$ a.s. $\forall T \in T$ predictable.

$\mathcal{A}^2 = \{\text{nondecreasing RCLL predictable proc. } A \text{ with } A_0 = 0 \text{ and } \mathbb{E}(A^2_T) < \infty\}$

if $\xi$ (resp. $-\zeta$) is left-u.s.c. along stopping times, then $A$ (resp. $A'$) is continuous.
Game options in the imperfect market $\mathcal{M}^g$

Main results (2/2)

- When $\zeta$ is left l.s.c along stopping times (and $\xi$ only RCLL), the seller's price $u_0 := \inf\{x \in \mathbb{R}, \exists (\sigma, \varphi) \in S(x)\}$ is the super-hedging price ($\inf = \min$). Let

$$\sigma^* := \inf\{t \geq 0, \ Y_t = \zeta_t\} \quad \text{and} \quad \varphi^* := \Phi(Z, K).$$

The pair $(\sigma^*, \varphi^*)$ is a super-hedge for the initial capital $u_0$.

- When $\zeta$ only RCLL, may not exist a super-hedge. However $\forall \varepsilon > 0$ let

$$\sigma_\varepsilon := \inf\{t \geq 0 : Y_t \geq \zeta_t - \varepsilon\} \quad \text{and} \quad \varphi^* := \Phi(Z, K).$$

The pair $(\sigma_\varepsilon, \varphi^*)$ is an $\varepsilon$-super-hedge for the initial capital $u_0$, i.e.

$$V_{Y_0, \varphi^*} \geq \xi_t, \quad 0 \leq t \leq \sigma_\varepsilon \quad \text{a.s.} \quad \text{and} \quad V_{\sigma_\varepsilon, \varphi^*} \geq \zeta_{\sigma_\varepsilon} - \varepsilon \quad \text{a.s.}$$

The seller is completely hedged before $\sigma_\varepsilon$ and hedged up to $\varepsilon$ at cancellation time $\sigma_\varepsilon$. 
Let $\mathcal{U}$ be a nonempty closed subset of $\mathbb{R}$ and $\mathcal{U}$ be the set of $U$-valued predictable processes.

To each $\alpha \in \mathcal{U}$ is associated a market model $\mathcal{M}_\alpha$, where the wealth process $V^{\alpha,x,\varphi}$ associated with initial wealth $x$ and strategy $\varphi$ satisfies

$$-dV_t^{\alpha,x,\varphi} = G(t, V_t^{\alpha,x,\varphi}, \varphi_t \sigma_t, -\varphi_t^2, \alpha_t)dt - \varphi_t \sigma_t dW_t + \varphi_t^2 dM_t$$

where $G$ is uniformly $\lambda$-admissible and satisfies the monotonicity conditions.

In the market model $\mathcal{M}_\alpha$, the nonlinear pricing system is given by $\mathcal{E}g^\alpha$, associated with driver $g^\alpha(t, \omega, y, z, k) := G(t, \omega, y, z, k, \alpha_t(\omega))$. 

Game options with model uncertainty

Robust superhedging

**Definition 1:** For given initial wealth $x$, a **robust super-hedge** is a pair $(\sigma, \varphi)$ of a stopping time $\sigma$ and a portfolio $\varphi$ such that for all $\alpha \in \mathcal{U}$, we have

$$V_{t}^{\alpha, x, \varphi} \geq \xi_{t}, \quad 0 \leq t \leq \sigma \text{ a.s. and } V_{\sigma}^{\alpha, x, \varphi} \geq \zeta_{\sigma} \text{ a.s.}$$

**Definition 2:** Define the **robust seller’s price** as

$$u_{0} := \inf \{ x \in \mathbb{R}, \exists (\sigma, \varphi) \in S^{r}(x) \},$$

where $S^{r}(x)$ is the set of all robust super-hedges associated with wealth $x$.

- if inf is **attained**, $u_{0}$ is called the **robust super-hedging price**.
Game options with model uncertainty

Dual problem

Let $\alpha \in \mathcal{U}$. The seller’s price of the game option in the market $\mathcal{M}_\alpha$ is characterized as its $g^\alpha$-value \( (= \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}_{0,\tau \wedge \sigma}[\mathcal{I}(\tau, \sigma)]) \)

Moreover, it is equal to $Y_0^\alpha$, where \((Y^\alpha, Z^\alpha, K^\alpha, A^\alpha, A'^\alpha)\) is the unique solution in $S^2 \times H^2 \times H^2_\lambda \times A^2 \times A^2$ of the Doubly Reflected BSDE associated with driver $g^\alpha$ and barriers $\xi$ and $\zeta$.

We introduce a dual problem associated to the seller’s super-hedging problem

\[
v_0 := \sup_{\alpha \in \mathcal{U}} Y_0^\alpha.
\]
Game options with model uncertainty

Main results (1/2)

The robust seller’s price $u_0 = \text{dual value function } v_0$. Steps:

1. $v_0 := \sup_{\alpha \in \mathcal{U}} Y_0^\alpha$ is the value of the mixed generalized Dynkin game:

$$v_0 = \sup_{\alpha \in \mathcal{U}} \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}_{0,\tau \wedge \sigma}^g[l(\tau, \sigma)] = \sup_{\alpha \in \mathcal{U}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathcal{E}_{0,\tau \wedge \sigma}^g[l(\tau, \sigma)]. \quad (1)$$

2. $v_0 = Y_0$, where $(Y, Z, K, A, A')$ is the solution of the DRBSDE associated with barriers $\xi$ and $\zeta$ and driver

$$g(t, \omega, y, z, k) := \sup_{\alpha \in \mathcal{U}} G(t, \omega, y, z, k, \alpha)$$

3. $u_0 = Y_0$.

Moreover, $\inf_{\sigma}$ and $\sup_{\alpha}$ can be interchanged in (1). So:

$$u_0 = \sup_{\alpha \in \mathcal{U}} \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}_{0,\tau \wedge \sigma}^g[l(\tau, \sigma)] = \inf_{\sigma \in \mathcal{T}} \sup_{\alpha \in \mathcal{U}} \sup_{\tau \in \mathcal{T}} \mathcal{E}_{0,\tau \wedge \sigma}^g[l(\tau, \sigma)].$$
When $\zeta$ is left l.s.c along stopping times (and $\xi$ only RCLL), the robust seller’s price $u_0$ is the robust super-hedging price ($\inf = \min$). Let $\sigma^* := \inf\{t \geq 0, \ Y_t = \zeta_t\}$ and $\varphi^* := \Phi(Z, K)$. The pair $(\sigma^*, \varphi^*)$ is a robust super-hedge for the initial capital $u_0$.

If $U$ compact, $\exists \bar{\alpha} \in U$ s.t. the robust superhedging price = the superhedging price in $M_{\bar{\alpha}}$, i.e. $u_0 = Y_{0\bar{\alpha}}$, and $\bar{\alpha}$ is a worst case scenario.

When $\zeta$ is only RCLL, there may not exist a robust super-hedge. However, $\forall \varepsilon > 0$, let $\sigma_\varepsilon := \inf\{t \geq 0 : \ Y_t \geq \zeta_t - \varepsilon\}$. The pair $(\sigma_\varepsilon, \varphi^*)$ is an $\varepsilon$-robust super-hedge, i.e. $\forall \alpha \in U$,

$$V_{t,\alpha,Y_0,\varphi^*} \geq \xi_t, \ 0 \leq t \leq \sigma_\varepsilon \ \text{a.s.} \ \text{and} \ V_{\sigma_\varepsilon,Y_0,\varphi^*} \geq \zeta_{\sigma_\varepsilon} - \varepsilon \ \text{a.s.}$$
Proof: based on properties (comparison thms, estimates, optimization principles...) on BSDEs and DRBSDEs and the characterization of the value of generalized Dynkin games in terms of nonlinear Doubly Reflected BSDEs (Dumitrescu-Quenez-Sulem, EJP(2016)).
Example with ambiguity on the default probability

Consider a family of probability measures $Q^\alpha$, equivalent to $P$, which admits $Z_T^\alpha$ as density with respect to $P$, with

$$dZ_t^\alpha = Z_t^\alpha \gamma(t, \alpha_t) dM_t; \quad Z_0^\alpha = 1,$$

where $\gamma$ bounded and $\gamma(t, \alpha) > C_1 > -1$.

Under $Q^\alpha$, $M_t^\alpha := N_t - \int_0^t \lambda_s (1 + \gamma(s, \alpha_s)) ds$ is a $\mathcal{G}$-martingale and $\gamma(t, \alpha_t)$ represents the uncertainty on the default intensity.

To each $\alpha$, corresponds a market model $\mathcal{M}_\alpha$ in which the wealth satisfies

$$-dV_t^{\alpha, x, \varphi} = f(t, V_t^{\alpha, x, \varphi}, \varphi'_t \sigma_t, -\varphi^2_t) dt - \varphi'_t \sigma_t dW_t + \varphi^2_t dM_t^\alpha; \quad V_0^{\alpha, x, \varphi} = x,$$

where $f$ is $\lambda$-admissible.

There exists an unique solution $(X^{\alpha}, Z^{\alpha}, K^{\alpha})$ of the following $Q^\alpha$-BSDE:

$$-dX_t^{\alpha} = f(t, X_t^{\alpha}, Z_t^{\alpha}, K_t^{\alpha}) dt - Z_t^{\alpha} dW_t - K_t^{\alpha} dM_t^\alpha; \quad X_T^{\alpha} = \xi.$$

The nonlinear price system $\mathcal{E}_Q^f$ in $\mathcal{M}_\alpha$, is thus the $f$-evaluation under $Q^\alpha$. 

(INRIA, Mathrisk)
Since $M_t^\alpha = M_t - \int_0^t \lambda_s \gamma(s, \alpha_s) ds$, the dynamics of the wealth in the market model $M_\alpha$ can be written as follows:

$$-dV_t^{\alpha, x, \varphi} = -\lambda_t \gamma(t, \alpha_t) \varphi_t^2 dt + f(t, V_t^{\alpha, x, \varphi}, \varphi_t \sigma_t, -\varphi_t^2) dt - \varphi_t \sigma_t dW_t + \varphi_t^2 dM_t.$$ 

This corresponds to our model with uncertainty where $G$ is given by:

$$G(t, \omega, y, z, k, \alpha) := \lambda_t(\omega) \gamma(t, \omega, \alpha) k + f(t, \omega, y, z, k).$$

**Proposition** The seller’s robust price of the game option in this model admits the following dual representation:

$$u_0 = \sup_{\alpha \in U} \inf_{\sigma \in T} \sup_{\tau \in T} \mathcal{E}^f_{Q^\alpha, 0, \tau \wedge \sigma}[I(\tau, \sigma)] = \inf_{\sigma \in T} \sup_{\alpha \in U} \sup_{\tau \in T} \mathcal{E}^f_{Q^\alpha, 0, \tau \wedge \sigma}[I(\tau, \sigma)].$$

Let $g$ be the map defined for each $(t, \omega, z, k)$ by

$$g(t, \omega, y, z, k) := \sup_{\alpha \in U} (\lambda_t(\omega) \gamma(t, \omega, \alpha) k + f(t, \omega, y, z, k)).$$

We have $u_0 = Y_0$, where $Y$ is the solution of the $P$-DRBSDE associated with driver $g$ and barriers $\xi$ and $\zeta$. 

*(INRIA, Mathrisk)*
Buyer’s point of view

European option

Consider a European option with maturity $T$ and payoff $\xi \in L^2(G_T)$. The buyer’s price of the option is equal to the opposite of the seller’s price of the option with payoff $-\xi$:

$$\tilde{E}_{g, S}(\xi) := -E_{g, S}(-\xi) = -X(T, -\xi).$$

Indeed, setting $\tilde{X}_0 := X_0(T, -\xi)$ and $\tilde{\phi} = \Phi(Z(T, -\xi), K(T, -\xi))$, we have $V_{T; \tilde{X}_0, \tilde{\phi}} + \xi = 0$ a.s.

Hence, if the initial price of the option is $-\tilde{X}_0$, he starts with $\tilde{X}_0$ at $t = 0$ and following the strategy $\tilde{\phi}$, the payoff he receives at $T$ allows him to recover the debt he incurred at $t = 0$ by buying the option.

The strategy $\tilde{\phi}$ is thus the hedging strategy for the buyer.

In the case of a perfect market, the dynamics of the wealth $X$ are linear wrt $(X, \phi)$, and the associated $g$-evaluation $\mathcal{E}^g$ is linear, so $\tilde{\mathcal{E}}^g = \mathcal{E}^g$. 
Buyer’s point of view - Game option

Supposing the initial price of the game option is $z$, the buyer starts with $-z$ at $t = 0$, and searches a super-hedge, i.e. an exercise time $\tau$ and a strategy $\varphi$, s.t. the payoff that he receives allows him to recover the debt he incurred at $t = 0$ by buying the game option, no matter the cancellation time chosen by the seller:

Definition 1

A buyer’s super-hedge against the game option with payoffs $(\xi, \zeta)$ and initial price $z \in \mathbb{R}$ is a pair $(\tau, \varphi)$ of a stopping time $\tau$ and a strategy $\varphi$ such that

$$V_{t}^{-z,\varphi} \geq -\zeta_{t}, \quad 0 \leq t < \tau \text{ a.s. and } V_{\tau}^{-z,\varphi} \geq -\xi_{\tau} \text{ a.s.}$$

Let $\mathcal{B}_{\xi,\zeta}(z)$: set of buyer’s super-hedges against the game option with payoffs $(\xi, \zeta)$ associated with initial price $z \in \mathbb{R}$.

The first inequality also holds at $t = \tau$ because $\xi \leq \zeta$. It follows that $\mathcal{B}_{\xi,\zeta}(z) = \mathcal{S}_{-\zeta,-\xi}(-z)$, where $\mathcal{S}_{-\zeta,-\xi}(-z)$ is the set of seller’s super-hedges against the game option with payoffs $(-\zeta, -\xi)$ associated with initial capital $-z$. 
Definition 2

The \textit{buyer’s price} \( \tilde{u}_0 \) of the game option is defined as the supremum of the initial prices which allow the buyer to be super-hedged, that is

\[
\tilde{u}_0 := \sup \{ z \in \mathbb{R}, \ \exists (\tau, \varphi) \in \mathcal{B}_{\xi, \zeta}(z) \}.
\]

Using the Remark, we derive that

\[
-\tilde{u}_0 = \inf \{ x \in \mathbb{R}, \ \exists (\tau, \varphi) \in \mathcal{S}_{-\zeta, -\xi}(x) \},
\]

which gives the following result:

Theorem 3

The buyer’s price of the game option with payoffs \((\xi, \zeta)\) is equal to the opposite of the seller’s price of the game option with payoffs \((-\zeta, -\xi)\).

\textit{In the special case of a perfect market, the buyer’s price is equal to the seller’s price.}
Definition 4

For a given initial wealth $z \in \mathbb{R}$, a buyer robust super-hedge against the game option with payoffs $(\xi, \zeta)$ is a pair $(\tau, \varphi)$ of a stopping time $\tau \in \mathcal{T}$ and a risky-assets strategy $\varphi \in \mathcal{H}_2^2 \times \mathcal{H}_\lambda^2$ such that

$$V_{t}^{\alpha, -z; \varphi} \geq -\zeta_t, \quad 0 \leq t < \tau \text{ a.s. and } V_{\tau}^{\alpha, -z; \varphi} \geq -\xi_{\tau} \text{ a.s., } \forall \alpha \in \mathcal{U}.$$ 

The buyer’s robust price of the game option is defined as the supremum of the initial prices which allow the buyer to construct a robust superhedge:

$$\tilde{u}_0 := \sup \{ z \in \mathbb{R}, \exists (\tau, \varphi) \in \mathcal{B}_{\xi, \zeta}^r(z) \}.$$ 

where $\mathcal{B}_{\xi, \zeta}^r(z)$ the set of all buyer’s robust super-hedges against the game option with payoffs $(\xi, \zeta)$ associated with initial price $z \in \mathbb{R}$.
Theorem 5

The buyer’s robust price of the game option with payoffs \((\xi, \zeta)\) is equal to the opposite of the seller’s robust price of the game option with payoffs \((-\zeta, -\xi)\).

We thus have the following dual formulation of the buyer’s robust price:

\[
\tilde{u}_0 = \inf_{\alpha \in \mathcal{U}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \tilde{E}_{\alpha, \tau \wedge \sigma}^g[I(\tau, \sigma)] = \inf_{\alpha \in \mathcal{U}} \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \tilde{E}_{\alpha, \tau \wedge \sigma}^g[I(\tau, \sigma)].
\]

The buyer’s robust price \(\tilde{u}_0\) is thus equal to the infimum over \(\alpha \in \mathcal{U}\) of the buyer’s prices in \(\mathcal{M}_\alpha\).
Conclusion/Summary

1. We have studied game options (pricing and superhedging issues) in a financial market with default and imperfections taken into account via the nonlinearity of the wealth dynamics. We proved that the seller’s price of a game option coincides with the value function of a corresponding generalized Dynkin game with $g$-evaluation. Links with associated DRBSDE also provide the $(\varepsilon)$-superhedging strategy.

2. We have also studied these issues in the case of model uncertainty, in particular ambiguity on the default probability, and characterize the seller’s price of a game option with model uncertainty as the value function of a mixed generalized Dynkin game.

Our approach relies on links between generalized (mixed) Dynkin games and doubly reflected BSDEs, and their properties.

Thank you for your attention!