Game options in an imperfect financial market with default and model uncertainty

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- Derivative contracts, introduced by Kifer in 2000, which can be terminated by both counterparties at any time before maturity T.
- extend the setup of American options by allowing the seller to cancel the contract

- If the buyer exercises at time τ, he gets ξ_τ from the seller,
- but if the seller cancels at σ before τ, he pays $\zeta_\sigma \geq \xi_\sigma$ to the buyer.

In short, if the buyer exercises at a stopping time $\tau \leq T$ and the seller cancels at a stopping time $\sigma \leq T$, then the seller pays to the buyer the payoff $\xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\tau > \sigma}$ at terminal time $\tau \wedge \sigma$.

The difference $\zeta_t - \xi_t$ for all t and is interpreted as a penalty for the seller for cancellation of the contract.

Game options

Kiefer's result

In the case of perfect markets, Kifer introduces the *fair price* u_0 of the game option, as the minimum initial wealth for the seller to cover his liability to pay the payoff to the buyer until a cancellation time, whatever the buyer's exercise time.

He shows both in the CRR discrete-time and Black-Scholes model (with ξ and ζ continuous), that u_0 is equal to the value function of a Dynkin game:

$$u_0 = \sup_{\tau} \inf_{\sigma} \mathbb{E}_Q[\tilde{\xi}_\tau \mathbf{1}_{\tau \leq \sigma} + \tilde{\zeta}_\sigma \mathbf{1}_{\tau > \sigma}] = \inf_{\sigma} \sup_{\tau} \mathbb{E}_Q[\tilde{\xi}_\tau \mathbf{1}_{\tau \leq \sigma} + \tilde{\zeta}_\sigma \mathbf{1}_{\tau > \sigma}],$$

where $\tilde{\xi}_t$ and $\tilde{\zeta}_t$ are the discounted values of ξ_t and ζ_t, and Q is the unique martingale probability measure.

Our goal

1. Study game options (pricing and superhedging) in the case of **imperfections in the market** taken into account via the *nonlinearity* of the wealth dynamics.
   We moreover include the possibility of a **default** and *irregular payoffs* (RCLL only).

2. Study game options **under model uncertainty**, in particular *ambiguity on the default probability*.

Financial market with default

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space.
Consider a market with three assets with price process $S = (S^0, S^1, S^2)$:

$$\begin{cases} dS^0_t = S^0_t r_t dt \\ dS^1_t = S^1_t[\mu^1_t dt + \sigma^1_t dW_t] \\ dS^2_t = S^2_{t^-}[\mu^2_t dt + \sigma^2_t dW_t - dM_t], \end{cases}$$

- W is a unidimensional standard Brownian motion
- $M_t = N_t - \int_0^t \lambda_s ds$ is the compensated martingale of the jump process $N_t := \mathbf{1}_{\vartheta \le t}$, $t \in [0, T]$, where ϑ is a r. v. modeling a default time. This default can appear at any time, i.e. $\mathbb{P}(\vartheta \ge t) > 0 \; \forall t \in [0, T]$.

Let $\mathbb{G} = \{\mathcal{G}_t, 0 \le t \le T\}$ the augmented filtration generated by W and N.
Suppose W is a $\mathbb{G}$-Brownian motion.
Process S^2 is the price of a defaultable asset with total default. Vanishes after ϑ.
$\sigma^1, \sigma^2, r, \mu^1, \mu^2$ predictable; $\sigma^1, \sigma^2 > 0$; r, σ^1, σ^2, μ^1, μ^2, λ, λ^{-1},$(\sigma^1)^{-1}$, $(\sigma^2)^{-1}$ bounded.

Option pricing in the perfect market case

Consider an investor with initial wealth x and risky asset strategy $\varphi = (\varphi^1, \varphi^2)$. Let $V_t^{x,\varphi}$ the value of the portfolio at time t.

Self financing condition:

$$dV_t = (r_t V_t + \varphi_t^1 \sigma_t^1 \theta_t^1 - \varphi_t^2 \theta_t^2 \lambda_t)dt + \varphi_t' \sigma_t dW_t - \varphi_t^2 dM_t,$$

where $\theta_t^1 := (\mu_t^1 - r_t)(\sigma_t^1)^{-1}$; $\theta_t^2 := -(\mu_t^2 - \sigma_t^2 \theta_t^1 - r_t)\lambda_t^{-1} \mathbf{1}_{\{t \leq \vartheta\}}$.

Consider a **European option** with maturity T and payoff ξ in $L^2(\mathcal{G}_T)$.

The unique solution $(X, Z, K) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ of the λ-**linear BSDE** (DQS'16)

$$-dX_t = -(r_t X_t + (Z_t + \sigma_t^2 K_t)\theta_t^1 + K_t \theta_t^2 \lambda_t)dt - Z_t dW_t - K_t dM_t\,;\; X_T = \xi.$$

provides the **replicating** portfolio : $\varphi_t{}' \sigma_t = Z_t$; $-\varphi_t^2 = K_t$.

This defines a change of variables:

$$\Phi(Z, K) := \varphi = (\varphi^1, \varphi^2) \text{ with } \varphi_t^2 = -K_t; \varphi_t^1 = (Z_t + \sigma_t^2 K_t)(\sigma_t^1)^{-1}.$$

$X = X(\xi)$ coincides with $V^{X_0,\varphi}$, the value of the (hedging) portfolio associated with initial wealth $x = X_0$ and portfolio strategy φ. We have:

$$X_t(\xi) = \mathbb{E}[e^{-\int_t^T r_s ds} \zeta_{t,T} \xi \,|\, \mathcal{G}_t],$$

where ζ satisfies

$$d\zeta_{t,s} = \zeta_{t,s^-}[-\theta_s^1 dW_s - \theta_s^2 dM_s];\ \zeta_{t,t} = 1,$$

with $\theta_t^1 := (\mu_t^1 - r_t)(\sigma_t^1)^{-1}$; $\theta_t^2 := -(\mu_t^2 - \sigma_t^2 \theta_t^1 - r_t)\lambda_t^{-1} \mathbf{1}_{\{t \leq \vartheta\}}$.

This defines a *linear* price system X: $\xi \mapsto X(\xi)$.

When $\theta_t^2 < 1$, $0 \leq t \leq \vartheta$ $dt \otimes dP$-a.s. Then $\zeta_{t,\cdot} > 0$.
The probability Q which admits $\zeta_{0,T}$ as density on $\mathcal{G}_T$, is the unique martingale probability measure.
In this case, the price system X is increasing and corresponds to the classical free-arbitrage price system.

The imperfect market model $\mathcal{M}^g$

Consider now the case of imperfections in the market taken into account via the *nonlinearity* of the dynamics of the wealth $V_t^{x,\varphi}$:

$$-dV_t = g(t, V_t, {\varphi_t}'\sigma_t, -\varphi_t^2)dt - {\varphi_t}'\sigma_t dW_t + \varphi_t^2 dM_t, \; V_0 = x$$

or equivalently, setting $Z_t = {\varphi_t}'\sigma_t$ and $K_t = -\varphi_t^2$,

$$-dV_t = g(t, V_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t.$$

In a perfect market, $g(t, x, z, k) = -r_t x - (z + \sigma_t^2 k)\theta_t^1 - \theta_t^2 \lambda_t k.$

Here g is a *nonlinear λ-admissible driver*, i.e. measurable, $g(., 0, 0, 0) \in \mathbb{H}^2$,

$$|g(\omega, t, y_1, z_1, k_1) - g(\omega, t, y_2, z_2, k_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + \sqrt{\lambda}_t |k_1 - k_2|).$$

- *Different borrowing and lending interest rates* R_t and r_t with $R_t \geq r_t$

$$g(t, V_t, \varphi'_t\sigma_t, -\varphi^2_t) := -(r_t V_t + \varphi^1_t\theta^1_t\sigma^1_t - \varphi^2_t\lambda_t\theta^2_t) + (R_t - r_t)(V_t - \varphi^1_t - \varphi^2_t)^-,$$

where φ^2_t vanishes after ϑ.

- *Large investor seller* whose trading strategy φ_t affects the market

$$g(t, V_t, \varphi_t\sigma_t, -\varphi^2_t) := -\bar{r}(t, V_t, \varphi_t)V_t - \varphi^1_t\,(\bar{\theta}^1\bar{\sigma}^1)(t, V_t, \varphi_t) + \varphi^2_t\lambda_t\,\bar{\theta}^2(t, V_t, \varphi_t).$$

- *Taxes on risky investments profits*

$$g(t, V_t, \varphi_t\sigma_t, -\varphi^2_t) := -(r_t V_t + \varphi^1_t\theta^1_t\sigma^1_t - \varphi^2_t\theta^2_t\lambda_t) + \rho(\varphi^1_t + \varphi^2_t)^+.$$

Here, $\rho \in \,]0, 1[$ represents an instantaneous tax coefficient.

Nonlinear pricing in the imperfect market $\mathcal{M}^g$

Consider a European option with maturity T and terminal payoff $\xi \in L^2(\mathcal{G}_T)$. $\exists!$ (X, Z, K) in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ solution of the BSDE

$$-dX_t = g(t, X_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t; \quad X_T = \xi.$$

The process X is equal to the wealth process associated with initial value $x = X_0$ and strategy $\varphi = \Phi(Z, K)$, that is $X = V^{X_0,\varphi}$.

Its initial value $X_0 = X_0(T, \xi)$ is thus a sensible price at time 0 for the seller of the claim ξ since it allows him to construct a hedging strategy φ s.t. the value of the associated portfolio is equal to ξ at time T. Similarly for $X_t = X_t(T, \xi)$ and it is the unique price which satisfying the hedging property.

This leads to a *nonlinear pricing* system, first introduced in NEK-Quenez'96 in Brownian framework, later called g*-evaluation* and denoted by $\mathcal{E}^g$:
$\forall S \in [0, T]$, $\forall \xi \in L^2(\mathcal{G}_S)$

$$\mathcal{E}^g_{t,S}(\xi) := X_t(S, \xi), t \in [0, S].$$

To ensure (strict) monotonicity and the no arbitrage property of the nonlinear pricing system $\mathcal{E}^g$, we assume

$$g(t,x,z,k_1) - g(t,x,z,k_2) \geq \gamma_t^{x,z,k_1,k_2}(k_1 - k_2)\lambda_t,$$

with $\gamma_t^{y,z,k_1,k_2} > -1$.

This is satisfied e.g. if g is non-decreasing wrt k, or if g is $\mathcal{C}^1$ in k with $\partial_k g(t,\cdot) > -\lambda_t$. on $\{t \leq \vartheta\}$.

In the special case of perfect market, $\partial_k g(t,\cdot) = -\theta_t^2$, and this Assumption is equivalent to $\theta_t^2 < 1$, the usual assumption.

Game options in the imperfect market $\mathcal{M}^g$

Definition

The game option consists for the seller to select a cancellation time $\sigma \in \mathcal{T}$ and for the buyer an exercise time $\tau \in \mathcal{T}$, so that the seller pays to the buyer at time $\tau \wedge \sigma$ the payoff

$$I(\tau, \sigma) := \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau}.$$

Assumptions: ξ, ζ : adapted RCLL processes in $\mathcal{S}^2$ with $\zeta_T = \xi_T$; $\xi_t \leq \zeta_t, 0 \leq t \leq T$ a.s. satisfying **Mokobodzki's condition:**
$\exists$ two nonnegative RCLL supermartingales H and H' in $\mathcal{S}^2$ such that:

$$\xi_t \leq H_t - H'_t \leq \zeta_t \quad 0 \leq t \leq T \quad \text{a.s.}$$

(holds e.g. when ξ or ζ is a semimartingale satisfying some integrability conditions).

Game options in the imperfect market $\mathcal{M}^g$

Definition 1: For each initial wealth x, a **super-hedge** against the game option is a pair of stopping time σ and portfolio strategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ s.t.

$$V_t^{x,\varphi} \geq \xi_t,\ 0 \leq t \leq \sigma \text{ a.s. and } V_\sigma^{x,\varphi} \geq \zeta_\sigma \text{ a.s.}$$

Definition 2: Define the **seller's price** of the game option as

$$u_0 := \inf\{x \in \mathbb{R},\ \exists(\sigma,\varphi) \in \mathcal{S}(x)\},$$

where $\mathcal{S}(x)$ is the set of all super-hedges associated with initial wealth x.
If inf is attained, u_0 is called the super-hedging price.

Definition 3: We define the **g-value** of the game option as

$$\inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)].$$

Game options in the imperfect market $\mathcal{M}^g$

Main results (1/2)

The seller's price u_0 of the game option is equal to the g-value.

Steps:

1. The g*-value* is equal to the value of the *generalized Dynkin game*

$$\inf_{\sigma\in\mathcal{T}}\sup_{\tau\in\mathcal{T}}\mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)]=\sup_{\tau\in\mathcal{T}}\inf_{\sigma\in\mathcal{T}}\mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)].$$

2. This value is equal to Y_0, where (Y,Z,K,A,A') is the unique solution in $\mathcal{S}^2\times\mathbb{H}^2\times\mathbb{H}^2_\lambda\times\mathcal{A}^2\times\mathcal{A}^2$ of the Doubly Reflected BSDE associated with barriers ξ and ζ and driver g.
3. $u_0=Y_0$.

Game options in the imperfect market $\mathcal{M}^g$

Associated Doubly Reflected BSDE

$$-dY_t = g(t, Y_t, Z_t, K_t)dt + dA_t - dA'_t - Z_t dW_t - K_t dM_t;\ Y_T = \xi_T,$$

$$(i)\ \xi_t \le Y_t \le \zeta_t,\ 0 \le t \le T \text{ a.s.},$$

$$(ii)\ dA_t \perp dA'_t \quad \textit{(the measures are mutually singular)}$$

$$(iii) \int_0^T (Y_t - \xi_t) dA^c_t = 0 \text{ a.s. and } \int_0^T (\zeta_t - Y_t) dA'^c_t = 0 \text{ a.s.}$$

$$\Delta A^d_\tau = \Delta A^d_\tau \mathbf{1}_{\{Y_{\tau-} = \xi_{\tau-}\}} \text{ and } \Delta A'^d_\tau = \Delta A'^d_\tau \mathbf{1}_{\{Y_{\tau-} = \zeta_{\tau-}\}} \text{a.s. } \forall \tau \in \mathcal{T} \text{pred}$$

$\mathcal{A}^2 = \{$nondecreasing RCLL predictable proc. A with $A_0 = 0$ and $\mathbb{E}(A_T^2) < \infty\}$

if ξ (resp. $-\zeta$) is left-u.s.c. along stopping times, then A (resp. A') is continuous.

Game options in the imperfect market $\mathcal{M}^g$

Main results (2/2)

- When ζ is left l.s.c along stopping times (and ξ only RCLL), the seller's price $u_0 := \inf\{x \in \mathbb{R},\ \exists(\sigma, \varphi) \in \mathcal{S}(x)\}$ is the *super-hedging* price $(\inf = \min)$. Let

$$\sigma^* := \inf\{t \geq 0,\ Y_t = \zeta_t\} \text{ and } \varphi^* := \Phi(Z, K).$$

The pair (σ^*, φ^*) is a super-hedge for the initial capital u_0.

- When ζ only RCLL, may not exist a *super-hedge*. However $\forall \varepsilon > 0$ let

$$\sigma_\varepsilon := \inf\{t \geq 0 :\ \ Y_t \geq \zeta_t - \varepsilon\} \text{ and } \varphi^* := \Phi(Z, K).$$

The pair $(\sigma_\varepsilon, \varphi^*)$ is an ε-*super-hedge* for the initial capital u_0, i.e.

$$V_t^{Y_0,\varphi^*} \geq \xi_t,\ 0 \leq t \leq \sigma_\varepsilon \text{ a.s.} \quad \text{and} \quad V_{\sigma_\varepsilon}^{Y_0,\varphi^*} \geq \zeta_{\sigma_\varepsilon} - \varepsilon \text{ a.s.}$$

The seller is completely hedged before σ_ε and hedged up to ε at cancellation time σ_ε

Let U be a nonempty closed subset of $\mathbb{R}$ and $\mathcal{U}$ be the set of U-valued predictable processes.

To each $\alpha \in \mathcal{U}$ is associated a market model $\mathcal{M}_\alpha$, where the wealth process $V^{\alpha,x,\varphi}$ associated with initial wealth x and stategy φ satisfies

$$-dV_t^{\alpha,x,\varphi} = G(t, V_t^{\alpha,x,\varphi}, \varphi_t\sigma_t, -\varphi_t^2, \alpha_t)dt - \varphi_t\sigma_t dW_t + \varphi_t^2 dM_t$$

where G is *uniformly* λ*- admissible* and satisfies the monotonicity conditions.

In the market model $\mathcal{M}_\alpha$, the nonlinear pricing system is given by $\mathcal{E}^{g^\alpha}$, associated with driver $g^\alpha(t,\omega,y,z,k) := G(t,\omega,y,z,k,\alpha_t(\omega))$.

Definition 1: For given initial wealth x, a **robust super-hedge** is a pair (σ, φ) of a stopping time σ and a portfolio φ such that **for all** $\alpha \in \mathcal{U}$, we have

$$V_t^{\alpha,x,\varphi} \geq \xi_t,\ 0 \leq t \leq \sigma \ \text{ a.s. and } V_\sigma^{\alpha,x,\varphi} \geq \zeta_\sigma \text{ a.s.}$$

Definition 2: Define the *robust seller's price* as

$$\mathbf{u_0} := \inf\{x \in \mathbb{R},\ \exists(\sigma, \varphi) \in \mathcal{S}^r(x)\},$$

where $\mathcal{S}^r(x)$ is the *set of all robust super-hedges associated with wealth* x.

- if inf is **attained**, u_0 is called the **robust super-hedging price**.

Let $\alpha \in \mathcal{U}$. The seller's price of the game option in the market $\mathcal{M}_\alpha$ is characterized as its g^α-*value* $(= \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)])$
Moreover, it is equal to Y_0^α, where $(Y^\alpha, Z^\alpha, K^\alpha, A^\alpha, A'^\alpha)$ is the unique solution in $S^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathcal{A}^2$ of the Doubly Reflected BSDE associated with driver g^α and barriers ξ and ζ.

We introduce a *dual problem* associated to the seller's super-hedging problem

$$v_0 := \sup_{\alpha \in \mathcal{U}} Y_0^\alpha.$$

The robust seller's price $\mathbf{u_0}$ = dual value function v_0.

Steps:

1. $v_0 := \sup_{\alpha \in \mathcal{U}} Y_0^\alpha$ is the value of the *mixed* generalized Dynkin game:

$$v_0 = \sup_{\alpha \in \mathcal{U}} \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \sup_{\alpha \in \mathcal{U}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)]. \quad (1)$$

2. $v_0 = Y_0$, where (Y, Z, K, A, A') is the solution of the DRBSDE associated with barriers ξ and ζ and driver

$$\mathbf{g}(t,\omega,y,z,k) := \sup_{\alpha \in U} G(t,\omega,y,z,k,\alpha)$$

3. $\mathbf{u_0} = Y_0$.

Moreover, $\inf_\sigma$ and $\sup_\alpha$ can be interchanged in (1). So:

$$\mathbf{u_0} = \sup_{\alpha \in \mathcal{U}} \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \inf_{\sigma \in \mathcal{T}} \sup_{\alpha \in \mathcal{U}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)].$$

Game options with model uncertainty

Main results (2/2)

- When ζ is left l.s.c along stopping times (and ξ only RCLL), the robust seller's price $\mathbf{u_0}$ is the robust *super-hedging* price $(\inf = \min)$. Let $\sigma^* := \inf\{t \geq 0, \; Y_t = \zeta_t\}$ and $\varphi^* := \Phi(Z, K)$. The pair (σ^*, φ^*) is a robust super-hedge for the initial capital $\mathbf{u_0}$.

  If U compact, $\exists \bar{\alpha} \in \mathcal{U}$ s.t. the *robust* superhedging price $=$ the superhedging price in $\mathcal{M}_{\bar{\alpha}}$, i.e. $\mathbf{u_0} = Y_0^{\bar{\alpha}}$, and $\bar{\alpha}$ is a *worst case scenario*.

- When ζ is only RCLL, there may not exist a robust *super-hedge*. However, $\forall \varepsilon > 0$, let $\sigma_\varepsilon := \inf\{t \geq 0 : \;\; Y_t \geq \zeta_t - \varepsilon\}$. The pair $(\sigma_\varepsilon, \varphi^*)$ is an ε-*robust super-hedge*, i.e. $\forall \alpha \in \mathcal{U}$,

$$V_t^{\alpha, Y_0, \varphi^*} \geq \xi_t, \; 0 \leq t \leq \sigma_\varepsilon \text{ a.s.} \quad \text{and} \quad V_{\sigma_\varepsilon}^{\alpha, Y_0, \varphi^*} \geq \zeta_{\sigma_\varepsilon} - \varepsilon \text{ a.s.}$$

Proof: based on properties (comparison thms, estimates, optimization principles...) on BSDEs and DRBSDEs and the characterization of the value of generalized Dynkin games in terms of nonlinear Doubly Reflected BSDEs (Dumitrescu-Quenez-Sulem, EJP(2016)).

Consider a family of probability measures Q^α, equivalent to P, which admits Z_T^α as density with respect to P, with

$$dZ_t^\alpha = Z_t^\alpha \gamma(t, \alpha_t) dM_t; \quad Z_0^\alpha = 1,$$

where γ bounded and $\gamma(t, \alpha) > C_1 > -1$.
Under Q^α, $M_t^\alpha := N_t - \int_0^t \lambda_s(1 + \gamma(s, \alpha_s))ds$ is a $\mathbb{G}$-martingale and $\gamma(t, \alpha_t)$ represents the *uncertainty on the default intensity*.

To each α, corresponds a market model $\mathcal{M}_\alpha$ in which the wealth satisfies

$$-dV_t^{\alpha,x,\varphi} = f(t, V_t^{\alpha,x,\varphi}, \varphi_t'\sigma_t, -\varphi_t^2)dt - \varphi_t'\sigma_t dW_t + \varphi_t^2 dM_t^\alpha; \;\; V_0^{\alpha,x,\varphi} = x,$$

where f is λ-*admissible*.
There exists an unique solution $(X^\alpha, Z^\alpha, K^\alpha)$ of the following Q^α-BSDE:

$$-dX_t^\alpha = f(t, X_t^\alpha, Z_t^\alpha, K_t^\alpha)dt - Z_t^\alpha dW_t - K_t^\alpha dM_t^\alpha; \quad X_T^\alpha = \xi.$$

The nonlinear price system $\mathcal{E}_{Q^\alpha}^f$ in $\mathcal{M}_\alpha$, is thus the f-evaluation under Q^α.

Since $M_t^\alpha = M_t - \int_0^t \lambda_s \gamma(s, \alpha_s) ds$, the dynamics of the wealth in the market model $\mathcal{M}_\alpha$ can be written as follows:

$$-dV_t^{\alpha,x,\varphi} = -\lambda_t \gamma(t, \alpha_t) \varphi_t^2 dt + f(t, V_t^{\alpha,x,\varphi}, \varphi_t \sigma_t, -\varphi_t^2) dt - \varphi_t \sigma_t dW_t + \varphi_t^2 dM_t.$$

This corresponds to our model with uncertainty where G is given by:

$$G(t, \omega, y, z, k, \alpha) := \lambda_t(\omega) \gamma(t, \omega, \alpha) k + f(t, \omega, y, z, k).$$

Proposition The *seller's robust price* of the game option in this model admits the following dual representation:

$$\mathbf{u_0} = \sup_{\alpha \in \mathcal{U}} \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^f_{Q^\alpha, 0, \tau \wedge \sigma}[I(\tau, \sigma)] = \inf_{\sigma \in \mathcal{T}} \sup_{\alpha \in \mathcal{U}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^f_{Q^\alpha, 0, \tau \wedge \sigma}[I(\tau, \sigma)].$$

Let $\mathbf{g}$ be the map defined for each (t, ω, z, k) by

$$\mathbf{g}(t, \omega, y, z, k) := \sup_{\alpha \in U} \left(\lambda_t(\omega) \gamma(t, \omega, \alpha) k + f(t, \omega, y, z, k) \right).$$

We have $\mathbf{u_0} = Y_0$, where Y is the solution of the P-DRBSDE associated with driver $\mathbf{g}$ and barriers ξ and ζ.

Buyer's point of view

European option

Consider a European option with maturity T and payoff $\xi \in L^2(\mathcal{G}_T)$. The buyer's price of the option is equal to the opposite of the seller's price of the option with payoff $-\xi$:

$$\tilde{\mathcal{E}}^g_{\cdot,S}(\xi) := -\mathcal{E}^g_{\cdot,S}(-\xi) = -X_\cdot(T, -\xi).$$

Indeed, setting $\tilde{X}_0 := X_0(T, -\xi)$ and $\tilde{\varphi} = \Phi(Z(T, -\xi), K(T, -\xi))$, we have $V_T^{\tilde{X}_0, \tilde{\varphi}} + \xi = 0$ a.s.

Hence, if the initial price of the option is $-\tilde{X}_0$, he starts with $\tilde{X}_0$ at $t = 0$ and following the strategy $\tilde{\varphi}$, the payoff he receives at T allows him to recover the debt he incurred at $t = 0$ by buying the option.
The strategy $\tilde{\varphi}$ is thus the hedging strategy for the buyer.

In the case of a perfect market, the dynamics of the wealth X are linear wrt (X, φ), and the associated g-evaluation $\mathcal{E}^g$ is linear, so $\tilde{\mathcal{E}}^g = \mathcal{E}^g$.

Buyer's point of view - Game option

Supposing the initial price of the game option is z, the buyer starts with $-z$ at $t=0$, and searches a *super-hedge*, i.e. an exercise time τ and a strategy φ, s.t. the payoff that he receives allows him to recover the debt he incurred at $t=0$ by buying the game option, no matter the cancellation time chosen by the seller:

Definition 1

A *buyer's super-hedge* against the game option with payoffs (ξ, ζ) and initial price $z \in \mathbb{R}$ is a pair (τ, φ) of a stopping time τ and a strategy φ such that

$$V_t^{-z,\varphi} \geq -\zeta_t,\ 0 \leq t < \tau \text{ a.s. and } V_\tau^{-z,\varphi} \geq -\xi_\tau \text{ a.s.}$$

Let $\mathcal{B}_{\xi,\zeta}(z)$: set of buyer's super-hedges against the game option with payoffs (ξ, ζ) associated with initial price $z \in \mathbb{R}$.

The first inequality also holds at $t = \tau$ because $\xi \leq \zeta$. It follows that $\mathcal{B}_{\xi,\zeta}(z) = \mathcal{S}_{-\zeta,-\xi}(-z)$, where $\mathcal{S}_{-\zeta,-\xi}(-z)$ is the set of seller's super-hedges against the game option with payoffs $(-\zeta, -\xi)$ associated with initial capital $-z$.

Definition 2

The *buyer's price* $\tilde{u}_0$ of the game option is defined as the supremum of the initial prices which allow the buyer to be super-hedged, that is

$$\tilde{u}_0 := \sup\{z \in \mathbb{R}, \;\; \exists(\tau, \varphi) \in \mathcal{B}_{\xi,\zeta}(z)\}.$$

Using the Remark, we derive that

$$-\tilde{u}_0 = \inf\{x \in \mathbb{R}, \;\; \exists(\tau, \varphi) \in \mathcal{S}_{-\zeta,-\xi}(x)\},$$

which gives the following result:

Theorem 3

The buyer's price of the game option with payoffs (ξ, ζ) is equal to the opposite of the seller's price of the game option with payoffs $(-\zeta, -\xi)$.

In the special case of a perfect market, the buyer's price is equal to the seller's price .

Definition 4

For a given initial wealth $z \in \mathbb{R}$, a *buyer robust super-hedge* against the game option with payoffs (ξ, ζ) is a pair (τ, φ) of a stopping time $\tau \in \mathcal{T}$ and a risky-assets strategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ such that

$$V_t^{\alpha,-z,\varphi} \geq -\zeta_t,\ 0 \leq t < \tau \ \text{ a.s. and } V_\tau^{\alpha,-z,\varphi} \geq -\xi_\tau \text{ a.s.}, \quad \forall \alpha \in \mathcal{U}.$$

The *buyer's robust price* of the game option is defined as the supremum of the initial prices which allow the buyer to construct a robust superhedge:

$$\tilde{\mathbf{u}}_\mathbf{0} := \sup\{z \in \mathbb{R},\ \ \exists(\tau, \varphi) \in \mathcal{B}^r_{\xi,\zeta}(z)\}.$$

where $\mathcal{B}^r_{\xi,\zeta}(z)$ the set of all buyer's robust super-hedges against the game option with payoffs (ξ, ζ) associated with initial price $z \in \mathbb{R}$.

Theorem 5

The buyer's robust price of the game option with payoffs (ξ, ζ) is equal to the opposite of the seller's robust price of the game option with payoffs $(-\zeta, -\xi)$.

We thus have the following dual formulation of the buyer's robust price:

$$\tilde{\mathbf{u}}_\mathbf{0} = \inf_{\alpha \in \mathcal{U}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \tilde{\mathcal{E}}^{g^\alpha}_{0,\tau \wedge \sigma}[I(\tau, \sigma)] = \inf_{\alpha \in \mathcal{U}} \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \tilde{\mathcal{E}}^{g^\alpha}_{0,\tau \wedge \sigma}[I(\tau, \sigma)].$$

The buyer's robust price $\tilde{\mathbf{u}}_\mathbf{0}$ is thus equal to the infimum over $\alpha \in \mathcal{U}$ of the buyer's prices in $\mathcal{M}_\alpha$.

Conclusion/Summary

1. We have studied game options (pricing and superhedging issues) in a financial market with default and imperfections taken into account via the *nonlinearity* of the wealth dynamics.
   We proved that the seller's price of a game option coincides with the value function of a corresponding *generalized* Dynkin game with g-evaluation. Links with associated DRBSDE also provide the (ε)-superhedging strategy.

2. We have also studied these issues in the case of **model uncertainty**, in particular *ambiguity on the default probability*, and characterize the seller's price of a game option with model uncertainty as the value function of a *mixed generalized* Dynkin game.

Our approach relies on links between *generalized* (mixed) Dynkin games and doubly reflected BSDEs, and their properties.

Thank you for your attention !