Remarks on rough Bergomi: asymptotics and calibration

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Based on joint works with
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Contents

Motivation

- Classical stochastic volatility models generate a constant short-maturity ATM skew and a large-maturity one proportional to τ^{-1};
- However, short-term data suggests a time decay of the ATM skew proportional to $\tau^{-\alpha}$, $\alpha \in (0, 1/2)$.
- One solution: adding volatility factors (risk of over-parameterisation). Gatheral's Double Mean-Reverting, Bergomi-Guyon, each factor acting on a specific time horizon.
- In the Lévy case (Tankov, 2010), the situation is different, as $\tau \downarrow 0$:
  - in the pure jump case with $\int_{(-1,1)} |x|\nu(\mathrm{d}x) < \infty$, then $\sigma_\tau^2(0) \sim c\tau$;
  - in the (α) stable case, $\sigma_\tau^2(0) \sim c\tau^{1-2/\alpha}$ for $\alpha \in (1, 2)$;
  - for out-of-the-money options, $\sigma_\tau^2(k) \sim \dfrac{k^2}{2\tau|\log(\tau)|}$.

Rough volatility models

- Gatheral-Jaisson-Rosenbaum (2014)–based on Comte-Coutin-Renault–proposed a fractional volatility model:

$$\begin{aligned} \mathrm{d}S_t &= \sigma_t S_t \mathrm{d}B_t, \\ \sigma_t &= \exp(Z_t), \end{aligned}$$

  where B is a standard Brownian motion, and Z a fractional OU process satisfying $\mathrm{d}Z_t = \kappa(\theta - Z_t)\mathrm{d}t + \nu \mathrm{d}W_t^H$.
- Time series of the Oxford-Man SPX realised variance as well as implied volatility smiles of the SPX suggest that $H \in (0, 1/2)$: **short-memory** volatility.
- Is not statistically rejected by Ait-Sahalia-Jacod's test (2009) for Itô diffusions.
- Main drawback: loss of Markovianity ($H \neq 1/2$) rules out PDE techniques, and Monte Carlo is computationally intensive. One way out is an efficient "Hybrid scheme" of Bennedsen, Lunde and Pakkanen (2015).

The Rough Bergomi model (Bayer-Friz-Gatheral)

Let Z be the process defined pathwise as

$$Z_t := \int_0^t K_\alpha(s,t)\mathrm{d}W_s, \qquad \text{for any } t \geq 0,$$

with $\alpha \in \left(-\frac{1}{2}, 0\right)$, W a standard Brownian motion, and the kernel K_α:

$$K_\alpha(u,s) := \eta\sqrt{2\alpha+1}(s-u)^\alpha, \qquad \text{for all } 0 \leq u \leq s,$$

for some strictly positive constant η. The rough Bergomi model is then defined as

$$\begin{aligned} X_t &= \int_0^t \sqrt{V_s}\mathrm{d}B_s - \frac{1}{2}\int_0^t V_s \mathrm{d}s, \qquad && X_0 = 0, \\ V_t &= V_0 \exp\left(Z_t - \frac{\eta^2}{2}t^{2\alpha+1}\right), && V_0 = 1, \end{aligned}$$

with $B := \rho W + \sqrt{1-\rho^2}W^\perp$, for $\rho \in (-1,1)$.

Comments on Rough Bergomi

Proposition

- $\exp(X)$ *is a true martingale.*
- *For any* $t \geq 0$, (Z_t, B_t) *is a centered Gaussian random variable with covariance*

$$\mathbb{E}(B_t Z_t) = \begin{pmatrix} \eta^2 t^{2\alpha+1} & \varrho t^{\alpha+1} \\ \varrho t^{\alpha+1} & t \end{pmatrix},$$

where $\varrho := \frac{\rho\eta\sqrt{2\alpha+1}}{\alpha+1}$, *and* (Z, B) *is Gaussian process. Furthermore*

$$\mathbb{E}(Z_s Z_t) = \frac{\eta^2(2\alpha+1)}{\alpha+1}(s \wedge t)^{1+\alpha}(s \vee t)^{\alpha} {}_2F_1\left(1, -\alpha, 2+\alpha, \frac{s \wedge t}{s \vee t}\right).$$

- $\log(V)$ *is almost surely locally* γ*-Hölder continuous, for all* $\gamma \in \left(0, \alpha + \frac{1}{2}\right)$ *[*$\alpha = H - 1/2$*].*

Remarks

- Z is self-similar;
- Z is the Holmgren-Riemann-Liouville fBm, not the standard (Mandelbrot-van Ness one), and is not stationary;
- Recall that for a standard fBm, for any $u \le t$,

$$\begin{aligned} W_t^H - W_u^H &= C_H \left\{ \int_u^t \frac{\mathrm{d}W_s}{(t-s)^{1/2-H}} + \int_{-\infty}^u \left[\frac{1}{(t-s)^{1/2-H}} - \frac{1}{(-s)^{1/2-H}} \right] \mathrm{d}W_s \right\} \\ &= Z_u(t) + G_u(t), \end{aligned}$$

where $G_u(t) \in \mathcal{F}_u^W$ whereas $Z_u(t) \perp \mathcal{F}_u^W$.

Contents

Quick reminder on (pathwise) Large Deviations

Let $\mathscr{E}$ denote a real, separable Banach Space with norm $\|\cdot\|_{\mathscr{E}}$, and $(\mu_\varepsilon)_{\varepsilon>0}$ a sequence of probability measures on $(\mathscr{E}, \mathscr{B}(\mathscr{E}))$.

Definition

The family $(\mu_\varepsilon)_{\varepsilon>0}$ satisfies a large deviations principle (LDP) as ε tends to zero with speed ε^{-1} and rate function Λ if, for any $B \in \mathscr{B}(\mathscr{E})$,

$$-\inf_{\mathfrak{z}\in B^\circ} \Lambda(\mathfrak{z}) \le \liminf_{\varepsilon\downarrow 0} \varepsilon \log(\mu_\varepsilon(B)) \le \limsup_{\varepsilon\downarrow 0} \varepsilon \log(\mu_\varepsilon(B)) \le -\inf_{\mathfrak{z}\in \overline{B}} \Lambda(\mathfrak{z}).$$

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Lighter versions:

- Take $\mathscr{E} = \mathbb{R}$, then LDP yields, for any $B \subset \mathbb{R}$,

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- Take $\mathscr{E} = \mathcal{C}$, the space of continuous paths. LDP yields, for any $B \subset \mathcal{C}$,

$$\mu_\varepsilon(B) \sim \exp\left\{-\frac{1}{\varepsilon} \inf_{\varphi\in B} \Lambda(\varphi)\right\}.$$

Asymptotic behaviour of Rough Bergomi

Rough Bergomi: $$X_t = \int_0^t \sqrt{V_s}\mathrm{d}B_s - \frac{1}{2}\int_0^t V_s \mathrm{d}s, \qquad V_t = V_0 \exp\left(Z_t - \frac{\eta^2}{2}t^{2\alpha+1}\right).$$

For $t, \varepsilon \geq 0$, define the rescaled random variables:

$$X_t^\varepsilon := \varepsilon^\beta X_{\varepsilon t}, \quad Z_t^\varepsilon := \varepsilon^{\beta/2} Z_t, \quad V_t^\varepsilon := \varepsilon^{1+\beta} \exp\left\{Z_t^\varepsilon - \frac{\eta^2}{2}(\varepsilon t)^\beta\right\}, \quad B_t^\varepsilon := \varepsilon^{\beta/2} B_t,$$

where $\beta := 2\alpha + 1 \in (0,1)$. Note that, for any $t, \varepsilon \geq 0$,

$$Z_t^\varepsilon \overset{(\text{law})}{=} Z_{\varepsilon t} \qquad \text{and} \qquad V_t^\varepsilon \overset{(\text{law})}{=} \varepsilon^{1+\beta} V_{\varepsilon t},$$

so that, for any $t \geq 0$,

$$X_t^\varepsilon = \int_0^t \sqrt{V_s^\varepsilon}\mathrm{d}B_s^\varepsilon - \frac{1}{2}\int_0^t V_s^\varepsilon \mathrm{d}s.$$

Main result: Large deviations

Theorem (Jacquier-Pakkanen-Stone)

The sequence $(X^{\varepsilon}_{\cdot})_{\varepsilon \geq 0}$ satisfies a LDP with speed $\varepsilon^{-\beta}$ and rate function

$$\Lambda^X(\varphi) := \inf \left\{ \Lambda(x, y) : \varphi = \int_0^{\cdot} x(s) \mathrm{d}y(s), y \in \mathrm{BV} \cap \mathcal{C} \right\}.$$

Define the operators (on $\mathcal{C}^2$ and $\mathcal{C}$ respectively)

$$\mathcal{M} \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \end{pmatrix} (t, \varepsilon) := \begin{pmatrix} (\mathfrak{m}\mathrm{x})(t, \varepsilon) \\ \mathrm{y}(t) \end{pmatrix} \quad \text{and} \quad (\mathfrak{m}\mathrm{x})(t, \varepsilon) := \varepsilon^{1+\beta} \exp \left(\varepsilon^{\beta/2} \mathrm{x}(t) - \frac{\eta^2}{2} (\varepsilon t)^{\beta} \right),$$

as well as the function $\Lambda : \mathcal{C}([0, 1]^2, \mathbb{R}_+ \times \mathbb{R}) \to \mathbb{R}_+$ by

$$\Lambda : \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \end{pmatrix} \mapsto \inf \left\{ \widetilde{\Lambda}(\overline{\mathrm{x}}, \overline{\mathrm{y}}) : \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \end{pmatrix} = \mathcal{M} \begin{pmatrix} \overline{\mathrm{x}} \\ \overline{\mathrm{y}} \end{pmatrix} \right\},$$

where $\widetilde{\Lambda} \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \end{pmatrix} := \frac{1}{2} \left\| \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \end{pmatrix} \right\|_{\mathscr{H}}^2$, and $\mathscr{H}$ is the RKHS of the measure induced by (Z, B).

Corollaries

Corollary (Small-time behaviour)

The process $\left(t^{\beta} X_t\right)_{t \geq 0}$ satisfies a LDP on $\mathbb{R}$ with speed $t^{-\beta}$ and rate function Λ^X.

Proof: By self-similarity.

Corollary (Implied volatility)

The following holds for all $x \neq 0$ ($\beta \in (0,1)$):

$$\lim_{t \downarrow 0} t^{1+\beta} \widehat{\sigma}\left(x t^{-\beta}, t\right)^2 = \frac{1}{2} \frac{|x|^2}{\inf_{y \geq x} \Lambda^X(y)}.$$

Proof Part 1: Reproducing Kernel Hilbert Space

Let $(\mathscr{E}, \|\cdot\|_{\mathscr{E}})$ be a real, separable Banach Space, and $\mathscr{E}^*$ its topological dual, with duality relationship $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{\mathscr{E}^*\mathscr{E}}$. For a Gaussian measure μ on $\mathscr{E}$, introduce the bounded, linear operator $\Gamma : \mathscr{E}^* \to \mathscr{E}$ as $\Gamma(f^*) := \int_{\mathscr{E}} \langle f^*, f\rangle f \mu(\mathrm{d}f)$.

Definition

The reproducing kernel Hilbert space (RKHS) $\mathscr{H}_\mu$ of μ is defined as the completion of $\Gamma(\mathscr{E}^*)$ with the inner product $\langle \Gamma(f^*), \Gamma(g^*)\rangle_{\mathscr{H}_\mu} := \int_{\mathscr{E}} \langle f^*, f\rangle\langle g^*, f\rangle_{\mathscr{E}^*\mathscr{E}}\mu(\mathrm{d}f)$.

Proposition

The RKHS of the induced measure (on $\mathcal{C}^2$) of the two-dimensional process (Z, B) is

$$\mathscr{H} = \left\{\left(\int_0^\cdot K_\alpha(u,\cdot)f(u)\mathrm{d}u, \int_0^\cdot \rho f(u)\mathrm{d}u\right) : f \in \mathrm{L}^2\right\},$$

with inner product

$$\left\langle \begin{pmatrix} \int_0^\cdot K_\alpha(u,\cdot)f_1(u)\mathrm{d}u \\ \rho\int_0^\cdot f_1(u)\mathrm{d}u \end{pmatrix}, \begin{pmatrix} \int_0^\cdot K_\alpha(u,\cdot)f_2(u)\mathrm{d}u \\ \rho\int_0^\cdot f_2(u)\mathrm{d}u \end{pmatrix} \right\rangle_{\mathscr{H}} := \langle f_1, f_2\rangle_{\mathrm{L}^2}.$$

Proof Part 2: Contraction mappings

- Following Deuschel-Stroock (for Gaussian measures), the sequence $(Z^\varepsilon, B^\varepsilon)_{\varepsilon>0}$ satisfies a LDP with speed $\varepsilon^{-\beta}$ and rate function

$$\Lambda^*(\mathrm{z}_y^x) = \begin{cases} \frac{1}{2}\left\|\mathrm{z}_y^x\right\|_{\mathscr{H}}^2, & \text{if } \mathrm{z}_y^x \in \mathscr{H}, \\ +\infty, & \text{otherwise.} \end{cases}$$

- Pathwise, we view $t \mapsto (Z_t^\varepsilon, B_t^\varepsilon)^\top$ as an element of $\mathcal{C}^2$; $\begin{pmatrix} v_t^\varepsilon \\ B_t^\varepsilon \end{pmatrix} = \mathcal{M}\begin{pmatrix} Z^\varepsilon \\ B^\varepsilon \end{pmatrix}(t,\varepsilon)$. $\mathcal{M}$ is a continuous operator with respect to the $\mathcal{C}(\mathcal{T}^2, \mathbb{R}_+ \times \mathbb{R})$ norm $\|\cdot\|_\infty$.

Proof Part 3: LDP for stochastic integrals

- Claim: the sequence $(\mathbb{I}(v^\varepsilon, B^\varepsilon))_{\varepsilon\geq 0} := (\int_0^\cdot \sqrt{v_s^\varepsilon}\mathrm{d}B_s^\varepsilon)_{\varepsilon\geq 0}$ satisfies a LDP.
  - $B^\varepsilon = \varepsilon^{\alpha+1/2}B$, so that
  $$\mathbb{I}(v^\varepsilon, B^\varepsilon) = \mathbb{I}(\varepsilon^{2\alpha}v^\varepsilon, \sqrt{\varepsilon}B)$$
  holds a.s.;
  - the sequence of (semi)-martingales $(\sqrt{\varepsilon}B)$ is uniformly exponentially tight;
  - the sequence $(\sqrt{\varepsilon^{2\alpha}v^\varepsilon})_{\varepsilon>0}$ is càdlàg, and $(\mathscr{F}_t)$-adapted;
  - Garcia's Theorem implies that $(\mathbb{I}(v^\varepsilon, B^\varepsilon))_{\varepsilon\geq 0}$ satisfies a LDP with speed $\varepsilon^{-(1+2\alpha)}$ and rate function
  $$\Lambda^X(\varphi) = \inf\left\{\Lambda(\mathrm{z}_y^x) : \varphi = \mathbb{I}(x,y), y \in \mathrm{BV}\cap\mathcal{C}\right\}.$$
- Final step: LDP for $X_\cdot^\varepsilon = \displaystyle\int_0^\cdot \sqrt{v_s^\varepsilon}\mathrm{d}B_s^\varepsilon - \frac{1}{2}\int_0^\cdot v_s^\varepsilon \mathrm{d}s$. For any $\delta > 0$,
$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log \mathbb{P}\left(|\mathbb{I}(v^\varepsilon, B^\varepsilon)(1) - X_1^\varepsilon| > \delta\right) = -\infty$$
and the theorem follows by exponential equivalence.

Contents

Rough Bergomi, version 2

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, \quad & X_0 = 0 \\ V_t &= \xi_0(t)\mathcal{E}(2\nu C_H \mathcal{V}_t), \quad & V_0 > 0, \end{aligned}$$

- The process $\mathcal{V}$, defined as

$$\mathcal{V}_t := \int_0^t (t-u)^{H_-}\mathrm{d}Z_u,$$

is a centred Gaussian process with covariance structure

$$\mathbb{E}(\mathcal{V}_t\mathcal{V}_s) = s^{2H}\int_0^1 \left(\frac{t}{s}-u\right)^{H_-}(1-u)^{H_-}\mathrm{d}u, \qquad \text{for any } s,t\in[0,1];$$

- $H_\pm := H \pm \frac{1}{2}$;
- $(\xi_0(t))_{t\geq 0}$ represents the initial forward variance curve: $\xi_0(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(t\sigma_0^2(t)\right)$, where $\sigma_0^2(t)$ is the fair strike of a variance swap with maturity t.

VIX Futures

- For a fixed maturity $T \geq 0$, define the VIX at time T via the continuous-time monitoring formula

$$\mathrm{VIX}_T^2 := \mathbb{E}\left(\frac{1}{\Delta} \int_T^{T+\Delta} \mathrm{d}\langle X_s, X_s \rangle \mathrm{d}s \middle| \mathcal{F}_T \right),$$

where Δ is equal to 30 days;

- Risk-neutral formula for the VIX future $\mathfrak{V}_T$ with maturity T is then given by

$$\mathfrak{V}_T := \mathbb{E}\left(\mathrm{VIX}_T | \mathcal{F}_0\right) = \mathbb{E}\left(\sqrt{\frac{1}{\Delta} \int_T^{T+\Delta} \xi_T(s) \mathrm{d}s} \middle| \mathcal{F}_0 \right);$$

- $\eta_T(t) := \exp\left(2\nu C_H \int_0^T (t-u)^{H-} \mathrm{d}Z_u\right) \in \mathcal{F}_T$ is lognormal, for $t \geq T$.
  This is the main challenge for simulation, and we use the hybrid scheme by Bennedsen-Lunde-Pakkanen (2016). However, since it is independent of ξ_0, robustness of simulation schemes for the VIX will not be affected by the qualitative properties of the initial forward variance curve.

VIX Futures: dynamics and bounds

Proposition

The VIX dynamics are given by

$$\mathrm{VIX}_T^2 = \frac{1}{\Delta}\int_T^{T+\Delta} \xi_0(t)\eta_T(t)\exp\left(\frac{\nu^2 C_H^2}{H}\left[(t-T)^{2H} - t^{2H}\right]\right)\mathrm{d}t,$$

and the forward variance curve ξ_T in the rBergomi model admits the representation

$$\xi_T(t) = \xi_0(t)\eta_T(t)\exp\left(\frac{\nu^2 C_H^2}{H}\left[(t-T)^{2H} - t^{2H}\right]\right), \qquad \text{for any } t \geq T.$$

Theorem

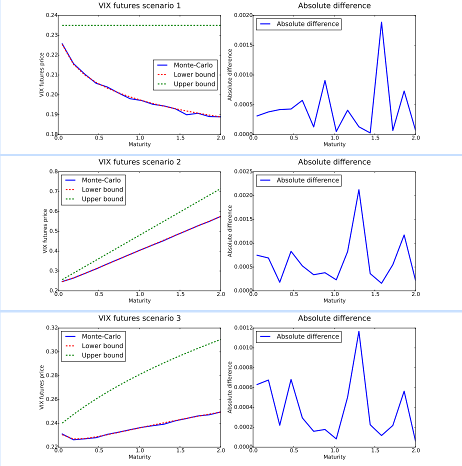
The following bounds hold for VIX Futures $\mathfrak{V}_T := \mathbb{E}\left(\mathrm{VIX}_T|\mathcal{F}_0\right)$:

$$\frac{1}{\Delta}\int_T^{T+\Delta} \sqrt{\xi_0(t)}\exp\left\{\frac{\nu^2 C_H^2}{4H}\left[(t-T)^{2H} - t^{2H}\right]\right\}\mathrm{d}t \leq \mathfrak{V}_T \leq \left\{\frac{1}{\Delta}\int_T^{T+\Delta}\xi_0(s)\mathrm{d}s\right\}^{\frac{1}{2}}.$$

Numerical remark

Scenarios for the initial forward variance curve:

$$[1]: \ \xi_0(t) = 0.234^2; \quad [2]: \ \xi_0(t) = 0.234^2(1+t)^2; \quad [3]: \ \xi_0(t) = 0.234^2\sqrt{1+t}.$$



Further properties of the VIX

Proposition

The following hold:

$$\sigma^2 := \mathbb{V}(\log(\Delta \mathrm{VIX}^2_{\mathcal{T}})) = -2\log\mathbb{E}(\Delta \mathrm{VIX}^2_{\mathcal{T}}) + \log\mathbb{E}[(\Delta \mathrm{VIX}^2_{\mathcal{T}})^2] =: -2\log\mathfrak{E}_1 + \log\mathfrak{E}_2,$$

$$\mu := \mathbb{E}(\log(\Delta \mathrm{VIX}^2_{\mathcal{T}})) = \log\mathfrak{E}_1 - \frac{\sigma^2}{2}.$$

with $\mathcal{T} := [T, T+\Delta]$, and

$$\mathfrak{E}_1 = \int_{\mathcal{T}} \xi_0(t)\mathrm{d}t,$$

$$\mathfrak{E}_2 = \int_{\mathcal{T}^2} \xi_0(u)\xi_0(t)\exp\left\{\frac{\nu^2 C_H^2}{H}\left[(u-T)^{2H} + (t-T)^{2H} - u^{2H} - t^{2H}\right]\right\} \mathrm{e}^{\overline{\Theta}_{u,t}}\mathrm{d}u\mathrm{d}t.$$

where $\overline{\Theta}_{u,t}$ is equal to zero if $u=t$ and otherwise equal to $\Theta_{u\vee t, u\wedge t}$, available in closed form in terms of the hypergeometric ${}_2F_1$ function.

Options on VIX

Assumption A: $\Delta \mathrm{VIX}_T^2$ is log-normal.

Proposition

- A VIX future is worth

$$\mathfrak{V}_T = \begin{cases} \sqrt{\frac{1}{\Delta}\int_T^{T+\Delta} \xi_0(t)\mathrm{d}t} \exp\left(-\frac{\sigma^2}{8}\right), & \text{under Assumption A,} \\ \sqrt{\frac{1}{\Delta}\int_T^{T+\Delta} \xi_0(t)\mathrm{d}t} \exp\left(-\frac{\tilde{\sigma}^2}{8}\right), & \text{in [BFG15].} \end{cases}$$

- For $0 \le t \le T$, let $\mathfrak{V}_T(t) := \mathbb{E}\left(\mathrm{VIX}_T | \mathcal{F}_t\right)$ denote the price at time t of a VIX future maturing at T. Under Assumption A,

$$\mathbb{E}\left[(\mathfrak{V}_T(T) - K)_+ | \mathcal{F}_0\right] = \sqrt{\frac{1}{\Delta}\int_T^{T+\Delta} \xi_0(t)\mathrm{d}t} \exp\left(-\frac{\sigma^2}{8}\right) \Phi(d_1) - K\Phi(d_2),$$

where $\widetilde{K} := \frac{1}{\sigma}[\log(K^2\Delta) - \log \int_T^{T+\Delta} \xi_0(t)\mathrm{d}t + \frac{\sigma^2}{2}]$, $d_1 := -\widetilde{K} + \frac{1}{2}\sigma$, $d_2 := -\widetilde{K}$.

Numerical tests: VIX Futures

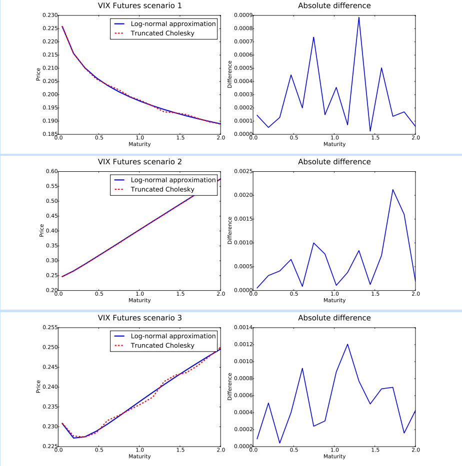


Figure: Log-normal approximations vs. simulations

VIX Futures Calibration

Calibration Goal:

$$\min_{\nu,H} \sum_{i=1}^{N} (\mathfrak{V}_{T_i} - \mathfrak{F}_i)^2,$$

where $(\mathfrak{F}_i)_{i=1,\dots,N}$ are the observed Futures prices on the time grid $T_1 < \dots < T_N$, $\mathfrak{V}_{T_i} = \sqrt{\frac{1}{\Delta}\int_{T_i}^{T_i+\Delta} \xi_0(t)\mathrm{d}t} \exp\left(-\frac{\sigma_i^2}{8}\right)$.

Obtaining the initial forward variance curve: ξ_0 depends on the current term structure of variance swaps, traded OTC. By replication, we calibrate a given implied volatility surface (eSSVI) and use it for interpolation/extrapolation:

$$\sigma^2_{\mathrm{BS}}(t,k)t := \frac{\theta_t}{2}\left\{1 + \rho(\theta_t)\varphi(\theta_t)k + \sqrt{(\varphi(\theta_t)k + \rho(\theta_t))^2 + 1 - \rho(\theta_t)^2}\right\},$$

$\theta_\cdot$: observed ATM variance curve; shape function: $\varphi(\theta) = \eta\theta^{-\lambda}(1+\theta)^{\lambda-1}$.
Correlation parameter:

$$\rho(\theta) = (A - C)\mathrm{e}^{-B\theta} + C, \qquad \text{for } (A, C) \in (-1,1)^2, B \geq 0,$$

ensuring that $|\rho(\cdot)| \leq 1$. Fair strike (in total variance) of a variance swap:

$$\sigma_0(t)^2 t := -2\mathbb{E}\log\left(\frac{S_t}{S_0}\right) = \frac{b_t^2 + 2a_t(c_t + \theta_t)}{2a_t^2},$$

and thus $\xi_0(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(t\sigma_0^2(t)\right) = \sigma_0^2(t) + t\frac{d}{\mathrm{d}t}\sigma_0^2(t)$.

Numerical results: SPX Fit

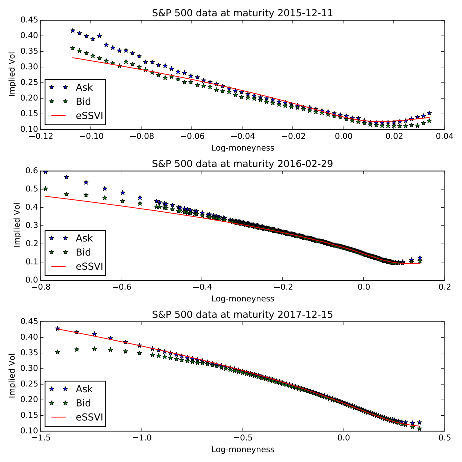


Figure: Calibration results on 4/12/2015 using traded SPX options.

VIX Futures calibration

Algorithm

(i) Calibrate eSSVI to available SPX option data;

(ii) compute the variance swap term structure $(\sigma_0(t)^2)_{t\geq 0}$;

(iii) extract the initial forward variance curve, $\xi_0(\cdot)$;

(iv) minimise (over ν, H) the objective function $\displaystyle\sum_{i=1}^{N}(\mathfrak{V}_{T_i} - \mathfrak{F}_i)^2$.

VIX Futures calibration

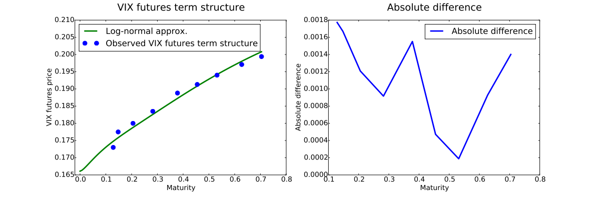


Figure: VIX Futures calibration on 4/12/2015. Optimal parameters: $(H, \nu) = (0.09237, 1.004)$.

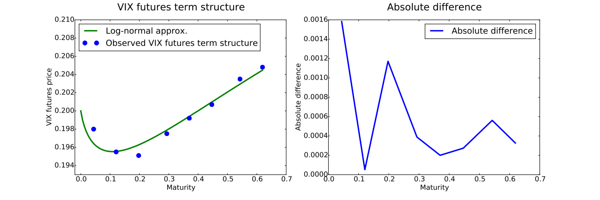


Figure: VIX Futures calibration on 4/1/2016. Optimal parameters: $(H, \nu) = (0.0509, 1.2937)$.

Is H consistent between VIX Futures and SPX?

We calibrate the model on 4/12/2015 by fixing $H = 0.09237$ obtained through VIX.

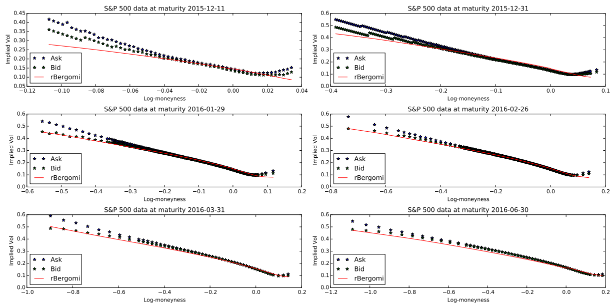


Figure: Calibration of SPX smiles on 4/12/2015. Calibrated parameters: $(\nu, \rho) = (1.19, -0.999)$.

Remark: *Regarding ν, we obtain a 20% difference between the one obtained through VIX calibration and the one obtained through SPX. This suggests that the volatility of volatility in the SPX market is 20% higher when compared to VIX. Nevertheless, we emphasise the importance of an accurate ξ_0 curve which could improve the fit to SPX and reduce the difference in ν to potentially unify a joint model.*

Elements of bibliography

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