Parameter estimation of Ornstein-Uhlenbeck process generating a stochastic graph

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Motivation

Model of interbank lending [CFS15, FI13]:

$$\mathrm{d}X_t^i = -\frac{a}{D}\sum_{j=0}^{D}\left(X_t^i - X_t^j\right)\mathrm{d}t + \sigma^i \mathrm{d}W_t^i$$

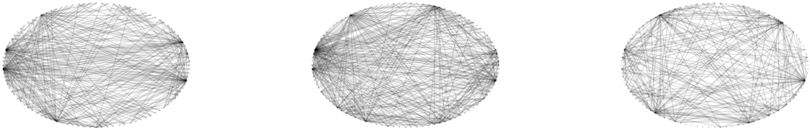
Generalization:

$$\mathrm{d}X_t = -AX_t\mathrm{d}t + \Sigma\mathrm{d}W_t$$

Ornstein-Uhlenbeck stochastic graphs

For measurable sets S define observations:

$$Y_t = \mathbb{1}_{X_t \in S} \qquad \text{e.g. } Y_t^{ij} = \mathbb{1}_{X_t^i - X_t^j \geq 1}$$



How to estimate $(A, \Sigma\Sigma^*)$ from the observation of Y?

1. High-frequency long-time limit: observations at $(k\Delta_n)_{0 \leq k \leq n}$, $\Delta_n \to 0$
2. Ergodicity, stationarity (stationary distribution $\mu_\infty = \mathcal{N}(0, V_\infty)$)
3. $a_0 := \min_{\lambda \in \mathrm{Sp}(A)} \mathcal{R}e(\lambda) > 0$

Convergence of statistics of Y

Theorem

Under $n\Delta_n \to \infty$:

$$\mathrm{OT}_n := \frac{1}{n}\sum_{k=0}^{n-1} Y_{k\Delta_n} \xrightarrow{L^2} \mu_\infty(S)$$

Also, in dimension 1, OT_n verifies a CLT property with speed $\sqrt{n\Delta_n}$.

Theorem

With $Y_t = \mathbb{1}_{X_t^1 \geq 1}$ and under $n\Delta_n^{3/2} \to \infty$:

$$\mathcal{C}_n := \frac{1}{n\sqrt{\Delta_n}}\sum_{k=0}^{n-1} \mathbb{1}_{Y_{k\Delta_n} \neq Y_{(k+1)\Delta_n}} \xrightarrow{L^2} 2\sqrt{\frac{(\Sigma\Sigma^*)^{11}}{2\pi}}\mu_{V_\infty^{11}}(1)$$

Application to parameter estimation

- Using OT_n estimate $\mu_\infty(S)$ for well-suited S, say $S^{ij} = \{X : X^i \geq 1, X^j \geq 1\}$.

$$\to V_\infty$$

- Using $\mathcal{C}_n$, estimate the diagonal of $\Sigma\Sigma^*$.

$$\to diag(\Sigma\Sigma^*)$$

- In the relevant case where A is diagonal, we can invert the relation between $A, \Sigma\Sigma^*, V_\infty$. We estimate A then use it to estimate $\Sigma\Sigma^*$.

$$\to A, \Sigma\Sigma^*$$

Elements of proof

Theorem (Gebelein inequality)

For H, K two closed subspaces of some Gaussian Hilbert space, define P_{HK} the restriction to H of the orthogonal projection onto K. Then

$$\sup_{A \in L^2(H), B \in L^2(K)} |\mathbb{C}\mathrm{or}\,(A, B)| = \|P_{HK}\|.$$

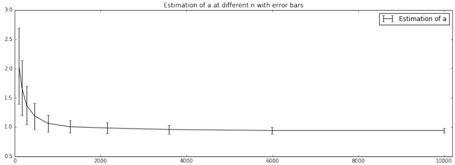
Theorem

For a suitable function g, denote $g_k^{(n)} = g(k, n, (X_s)_{k\Delta_n \leq s \leq (k+1)\Delta_n})$. Then, there is a finite constant $C_{(1)}$, dependent only on the parameters A, Σ of the model, such that:

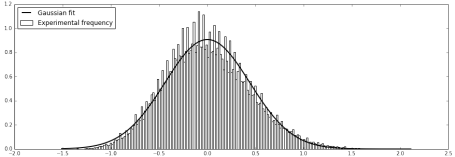
$$\mathbb{V}\mathrm{ar}\left(\sum_{k=0}^{n-1} \sqrt{\frac{\Delta_n}{n}} g_k^{(n)}\right) \leq C_{(1)} \sup_{k<n} \mathbb{V}\mathrm{ar}\left(g_k^{(n)}\right). \tag{1}$$

Numerical tests

- Estimation of a from model $\mathrm{d}X_t^i = -\frac{a}{D}\sum_{j=0}^{D}\left(X_t^i - X_t^j\right)\mathrm{d}t + \sigma^i \mathrm{d}W_t^i$

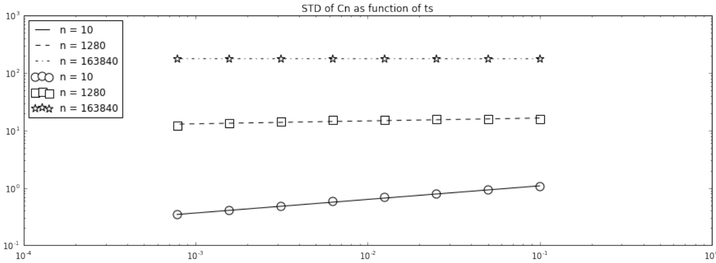


- CLT for $\mathcal{C}_n$?



Numerical tests

- Optimality of $\mathcal{C}_n$ convergence spped?



$$\sqrt{\mathbb{V}\mathrm{ar}\left(\mathcal{C}_n\right)} \propto \Delta_n^{-1} \neq \Delta_n^{-3/2}$$

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