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Hadrien De March

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Structure of the martinga plans in dimension 1

Potentials and irreducible components on R

Link between potential and convex functions in R

Structure of the martingale plans in higher dimension.

Definition of the irreducible convex paving in \mathbb{R}^d

Properties of the

Irreducible paving map for martingale couplings in finite dimension

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Existence of martingale couplings

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Definition of the irreducible convex paving in \mathbb{R}^d

Properties of the

- $\Omega = \mathbb{R}^d \times \mathbb{R}^d$ and X and Y are the two canonical random variables $\Omega \to \mathbb{R}^d$, $X : (x, y) \mapsto x$ and $Y : (x, y) \mapsto y$.
- The set of all martingale coupling probability laws between $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$:

$$\mathcal{M}(\mu,
u) := \{ \mathbb{P} \in \mathcal{P}(\Omega) : \mathbb{P} \circ X^{-1} = \mu, \mathbb{P} \circ Y^{-1} =
u$$

and $\mathbb{E}^{\mathbb{P}}[Y|X] = X \}.$

By (Strassen 1964), we have the following equivalence

 $\mathcal{M}(\mu,\nu) \neq \emptyset \iff \mu^{convex} \leq \nu$ i.e. $(\nu - \mu)[f] \geq 0$ for any f convex.

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Potentials and irreducible components on $\ensuremath{\mathbb{R}}$

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• Potential functions: powerful tools in dimension 1.

•
$$u_{\nu-\mu}(x) := \int_{\mathbb{R}} |t-x|(\nu-\mu)(dt).$$

- $u_{\nu-\mu} \ge 0.$
- For $\mathbb{P} \in \mathcal{M}(\mu, \nu)$,

$$u_{\nu-\mu}(x_0) = 0 \quad \Longleftrightarrow \quad \mathbb{E}^{\mathbb{P}}[|Y - x_0|] = \mathbb{E}^{\mathbb{P}}[|X - x_0|]$$
$$\iff \quad \mathbb{P}[Y > x_0|X \le x_0] = 0.$$

- Irreducible paving: $\{u_{\nu-\mu}(X)>0\} = \bigcup_{k\in\mathbb{N}}]a_k, b_k[.$
- Irreducible component *I_k* :=]*a_k*, *b_k*[(Beiglbock-Juillet 2012).
- $X \in I_k \implies Y \in \operatorname{cl} I_k, \ \mathcal{M}(\mu, \nu)$ -q.s.

Link between potential and convex functions in ${\mathbb R}$

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- $\mathfrak{C} := \{ f : \mathbb{R}^d \to \mathbb{R}, \text{ convex, and } (\nu \mu)[f] \leq 1 \}.$
- If d = 1, we have $(\nu \mu)[f] = \int_{\mathbb{R}} \frac{1}{2} u_{\nu \mu}(x) f''(x) dx$.
- $u_{\nu-\mu} > 0$ on I_k : \mathfrak{C} is compact when restricted to I_k for $f \in \mathfrak{C}$, as $(\nu \mu)[f] \leq 1$.
- To get compactness we need to "anchor" the convex function: f_k(x) = f(x) f(x_k) ∇f(x_k) · (x x_k) ≥ 0 for some x_k ∈ I_k.

• Idea of doubling the variable: $\mathbf{T}f(x, y) := f(y) - f(x) - \nabla f(x) \cdot (y - x) \ge 0.$

• Tangent convex functions: $\mathcal{T}(\mu, \nu) := \operatorname{cl} \mathsf{T}(\mathfrak{C})$

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Definition of the irreducible convex paving in \mathbb{R}^d

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• For
$$heta \in \mathcal{T}(\mu,
u)$$
, and $\mathbb{P} \in \mathcal{M}(\mu,
u)$:
 $\mathbb{E}^{\mathbb{P}}[\theta(X, Y)] \leq 1$.

- $Y \in \operatorname{dom} \theta(X, \cdot)$, $\mathcal{M}(\mu, \nu) q.s.$
- $\{\operatorname{dom}\theta(x,\cdot), x \in \mathbb{R}^d\}$ is a partition of \mathbb{R}^d .
- For K convex $G(K) := \dim(K) + g_K(K)$, with g_K Gaussian measure on Aff(K).
- *G* in increasing and bounded.
- Consider the minimization problem

Properties of the irreducible convex paving in \mathbb{R}^d

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Theorem

There is a μ -a.s. unique optimizer $I : \mathbb{R}^d \to \check{\mathcal{K}}$ to the optimization problem, called irreducible convex paving map. Moreover, we have the following properties: I is universally measurable, $\{I(x), x \in \mathbb{R}^d\}$ is a partition of \mathbb{R}^d with $X \in I(X)$, and

 $Y \in \operatorname{cl} I(X), \quad \mathcal{M}(\mu, \nu) - q.s.$

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Setwise duality

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Proposition

We may find a probability measure $\hat{\mathbb{P}} \in \mathcal{M}(\mu, \nu)$ and $\hat{\theta} \in \mathcal{T}(\mu, \nu)$ such that for any $\mathbb{P} \in \mathcal{M}(\mu, \nu)$ and $\theta \in \mathcal{T}(\mu, \nu)$,

$$\begin{split} \operatorname{supp} \mathbb{P}_X \subset & \operatorname{conv}(\operatorname{supp} \hat{\mathbb{P}}_X) \\ &= \operatorname{cl} I(X) \\ &= \operatorname{cl} \operatorname{dom} \hat{\theta}(X, \cdot) \subset & \operatorname{cl} \operatorname{dom} \theta(X, \cdot), \\ & \mu - a.s. \end{split}$$

Characterization of the $\mathcal{M}(\mu, \nu)$ -polar sets

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We also have the following characterization of $\mathcal{M}(\mu, \nu)$ -polar sets. We denote by \mathcal{N}_{μ} and \mathcal{N}_{ν} the collection of all negligible sets for μ and ν , respectively.

Proposition

A subset $N \in \mathcal{B}(\Omega)$, is $\mathcal{M}(\mu, \nu)$ -polar if and only if

$$N \subset \{X \in N_{\mu}\} \cup \{Y \in N_{\nu}\} \cup \{Y \notin J_{\theta}(X)\}$$

for some $N_{\mu} \in \mathcal{N}_{\mu}$, $N_{\nu} \in \mathcal{N}_{\nu}$, and $\theta \in \mathcal{T}(\mu, \nu)$.

Where $J_{\theta}(X) := I(X) \cup \operatorname{dom} \theta(X, \cdot) \cap \operatorname{cl} I(X)$.

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Questions ?



Figure: An example of Optimal Transport in practice.

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