Pricing and hedging with rough-Heston models

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Table of contents

Table of contents

A well-know stochastic volatility model

The Heston model

A very popular stochastic volatility model for a stock price is the Heston model :

$$dS_t = S_t\sqrt{V_t}dW_t$$

$$dV_t = \lambda(\theta - V_t)dt + \lambda\nu\sqrt{V_t}dB_t, \quad \langle dW_t, dB_t\rangle = \rho dt.$$

Popularity of the Heston model

- Reproduces several important features of low frequency price data : leverage effect, time-varying volatility, fat tails,...
- Provides quite reasonable dynamics for the volatility surface.
- Explicit formula for the characteristic function of the asset log-price$\rightarrow$ very efficient model calibration procedures.

But...

Volatility is rough !

- In Heston model, volatility follows a Brownian diffusion.
- It is shown in Gatheral *et al.* that log-volatility time series behave in fact like a fractional Brownian motion, with Hurst parameter of order 0.1.
- From Alos, Fukasawa and Bayer *et al.*, we know that such model also enables us to reproduce very well the behavior of the implied volatility surface, in particular the at-the-money skew (without jumps).

Rough-Heston model

Mandelbrot-van Ness representation

$$W_t^H = \int_0^t \frac{dW_s}{(t-s)^{\frac{1}{2}-H}} + \int_{-\infty}^0 \left(\frac{1}{(t-s)^{\frac{1}{2}-H}} - \frac{1}{(-s)^{\frac{1}{2}-H}}\right) dW_s.$$

Rough-Heston model

It is natural to modify Heston model and consider its rough version :

$$dS_t = S_t \sqrt{V_t} dW_t$$

$$V_t = V_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda(\theta - V_s) ds + \frac{\lambda\nu}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sqrt{V_s} dB_s,$$

with $\langle dW_t, dB_t \rangle = \rho dt$ and $\alpha \in (1/2, 1)$.

Pricing in Heston models

Classical Heston model

From simple arguments based on the Markovian structure of the model and Ito's formula, we get that in the classical Heston model, the characteristic function of the log-price $X_t = \log(S_t/S_0)$ satisfies

$$\mathbb{E}[e^{iaX_t}] = \exp\big(g(a,t) + V_0 h(a,t)\big),$$

where h is solution of the following Riccati equation :

$$\partial_t h = \frac{1}{2}(-a^2 - ia) + \lambda(ia\rho\nu - 1)h(a,s) + \frac{(\lambda\nu)^2}{2}h^2(a,s), \quad h(a,0) = 0,$$

and

$$g(a,t) = \theta\lambda \int_0^t h(a,s)ds.$$

Pricing in rough-Heston models

This work

- Goal : Deriving a Heston like formula in the rough case.
- Tool : The microstructural foundations of rough volatility models based on Hawkes processes.
- We build a sequence of relevant high frequency models converging to our rough-Heston process.
- We compute their characteristic function and pass to the limit.
- Application : Pricing and Hedging with rough-Heston models.

Table of contents

Price model

A tick by tick price model based on a two-dimensional Hawkes process (N_t^+, N_t^-) with intensity :

$$\begin{pmatrix}\lambda_t^+\\ \lambda_t^-\end{pmatrix} = \begin{pmatrix}\mu^+\\ \mu^-\end{pmatrix} + \int_0^t \begin{pmatrix}\varphi_1(t-s) & \varphi_3(t-s)\\ \varphi_2(t-s) & \varphi_4(t-s)\end{pmatrix} \cdot \begin{pmatrix}dN_s^+\\ dN_s^-\end{pmatrix},$$

where $\mu^+, \mu^- \in \mathbb{R}_+$ and $\varphi_1, \varphi_2, \varphi_3, \varphi_4 : \mathbb{R}_+ \to \mathbb{R}_+$

The microscopic price model

The price model :

$$P_t = N_t^+ - N_t^-$$

Encoding microscopic stylized facts

$$\begin{pmatrix}\lambda_t^+\\ \lambda_t^-\end{pmatrix} = \begin{pmatrix}\mu^+\\ \mu^-\end{pmatrix} + \int_0^t \begin{pmatrix}\varphi_1(t-s) & \varphi_3(t-s)\\ \varphi_2(t-s) & \varphi_4(t-s)\end{pmatrix} \cdot \begin{pmatrix}dN_s^+\\ dN_s^-\end{pmatrix},$$

Encoding microscopic stylized facts

- No arbitrage $\rightarrow \mu_+ = \mu_-$ and $\varphi_1 + \varphi_3 = \varphi_2 + \varphi_4$.
- Liquidity asymmetry on the bid and ask sides of the order book $\rightarrow \varphi_3 = \beta\varphi_2; \quad \beta > 1$.
- High endogeneity of markets $\rightarrow \rho(\int_0^\infty \phi) \approx 1$.
- Effect of metaorders $\rightarrow \varphi_1(x), \varphi_2(x) \sim K/x^{1+\alpha}$.

The scaling limit of the price model

Limit theorem

Under previous assumptions, and after suitable scaling in time and space, the long term limit of our price model satisfies the following rough-Heston log-price :

$$X_t = \int_0^t \sqrt{V_s} dW_s - \frac{1}{2}\int_0^t V_s ds,$$

$$V_t = V_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda(\theta - V_s)ds + \frac{\lambda\nu}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sqrt{V_s}dB_s,$$

with

$$d\langle W, B\rangle_t = \frac{1-\beta}{\sqrt{2(1+\beta^2)}}dt.$$

Characteristic function of multidimensional Hawkes processes

Using a population interpretation of Hawkes processes :

Theorem

The characteristic function of d-dimensional Hawkes process $(N_t)_{t\geq 0}$ with intensity :

$$\lambda_t = \mu(t) + \int_0^t \phi(t-s).dN_s \in \mathbb{R}^d.$$

is given by :

$$\mathbb{E}[\exp(ia.N_t)] = \exp\Big(\int_0^t \big(C(a,t-s) - \mathbf{1}\big).\mu(s)ds\Big),$$

where $C : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{C}^d$ is solution of the following integral equation :

$$C(a,t) = \exp\Big(ia + \int_0^t \phi^*(s).(C(a,t-s) - \mathbf{1})ds\Big).$$

Deriving the characteristic function

Strategy

- From our last theorem, we are able to derive the characteristic function of our high frequency price model.
- We then pass to the limit.

Characteristic function of rough-Heston models

We write :

$$I^{1-\alpha}f(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{f(t)}{(x-t)^\alpha}dt, \quad D^\alpha f(x) = \frac{d}{dx}I^{1-\alpha}f(x).$$

Theorem

The characteristic function at time t for the rough-Heston model is given by

$$\mathbb{E}[\exp\left(ia\log(S_t/S_0)\right)] = \exp\Big(\theta\lambda\int_0^t h(a,s)ds + V_0 I^{1-\alpha}h(a,t)\Big),$$

with $h(a,)$ the unique solution of the fractional Riccati equation :

$$D^\alpha h(a,s) = \frac{1}{2}(-a^2 - ia) + \lambda(ia\rho\nu - 1)h(a,s) + \frac{(\lambda\nu)^2}{2}h^2(a,s).$$

Table of contents

Characteristic function of rough-Heston models

Theorem

The characteristic function at time t for the rough-Heston model could be also written as

$$\mathbb{E}[\exp\left(ia\log(S_t/S_0)\right)] = \exp\Big(\int_0^t g(a,t-s)\mathbb{E}[V_s]ds\Big),$$

with :

$$g(a,t) = \frac{1}{2}(-a^2 - ia) + ia\rho\lambda\nu h(a,s) + \frac{(\lambda\nu)^2}{2}h^2(a,s).$$

This expression holds even for an extended rough-Heston model with time dependent parameter θ :

Extended rough-Heston model

$$dS_t = S_t\sqrt{V_t}dW_t$$

$$V_t = V_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda(\theta(s)-V_s)ds + \frac{\lambda\nu}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sqrt{V_s}dB_s.$$

Conditional law of the rough-Heston model

Theorem

The law of the process $(S_t^{t_0}, V_t^{t_0})_{t\geq 0} = (S_{t+t_0}, V_{t+t_0})_{t\geq 0}$ *is of a rough-Heston model with the following dynamics :*

$$dS_t^{t_0} = S_t^{t_0}\sqrt{V_t^{t_0}}dW_t^{t_0}; \quad S_0^{t_0} = S_{t_0}$$

$$V_t^{t_0} = V_{t_0} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda\big(\theta^{t_0}(s) - V_s^{t_0}\big)ds + \frac{\lambda\nu}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sqrt{V_s^{t_0}}dB_s^{t_0},$$

with $(W_t^{t_0}, B_t^{t_0}) = (W_{t_0+t} - W_{t_0}, B_{t_0+t} - B_{t_0})$ *is a correlated Brownian motion independent of* $\mathcal{F}_{t_0}$ *and* θ^{t_0} *is a* $\mathcal{F}_{t_0}$ *measurable process.*

Dynamics of the characteristic function process

Define :

$$P_t^T(a) = \mathbb{E}[\exp(ia\log(S_T))|\mathcal{F}_t]$$

Dynamics of the characteristic function process

$$P_t^T(a) = \exp\left(ia\log(S_t) + \int_0^{T-t} g(a,s)\mathbb{E}[V_{T-s}|\mathcal{F}_t]ds\right)$$

and

$$dP_t^T(a) = \partial_S P_t^T(a)dS_t + \partial_V P_t^T(a).(d\mathbb{E}[V_{T-s}|\mathcal{F}_t])_{0\leq s\leq T-t}$$

with :

$$\partial_S P_t^T(a) = ia\frac{P_t^T(a)}{S_t}; \quad \partial_V P_t^T(a) = P_t^T(a)(g(a,s))_{0\leq s\leq T-t}$$

We can hedge the option with the spot price and the forward variance curve !

Conclusion

About the rough-Heston model

- Consistent with historical data.
- Consistent with implied volatility surface data (In particular for the skew ATM).
- With a time dependent θ, the model is consistent with the forward variance curve of the market.
- Explicit formula of the characteristic function $\rightarrow$ fast calibration.
- Hedge formula using the underlying and the forward variance curve.
- But... Negative skew for the VIX : work in progress...