# Causal optimal transport and its links to enlargement of filtrations and stochastic optimization problems

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Q2: How to estimate the value of the additional information in terms of stochastic optimization problems (optimal value w.r.t. small & big filtration)?

$\rightarrow$ Both questions can be answered via causal optimal transport. Today we will concentrate on Q2.

# Optimal transport

**Monge-Kantorovich transport**: given two Polish probability spaces $(\mathcal{X}, \mu), (\mathcal{Y}, \nu)$, "move the mass" from $\mu$ to $\nu$ so as to minimize the cost of transportation $c : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \cup \{\infty\}$:

$$\inf \{\mathbb{E}^{\pi}[c(x, y)] : \pi \in \Pi(\mu, \nu)\},$$

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**Martingale transport**: $T_1$- and $T_2$-call prices $\Rightarrow S_{T_1}\sim\mu$, $S_{T_2}\sim\nu$. We “move $S_{T_1}$ to $S_{T_2}$” along a martingale. Robust price of a claim:

$$\inf\left\{\mathbb{E}^{\pi}[c(S_{T_1},S_{T_2})] : \pi\in\Pi(\mu,\nu),\ \pi \text{ is a martingale}\right\},\quad c=\text{payoff}.$$

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**Causal transport**: We will "move processes" $(X_t)_t\to(Y_t)_t$ along causal transport plans:

$$\inf\left\{\mathbb{E}^\pi[c(X,Y)]:\pi\in\Pi(\mu,\nu),\ \pi\text{ is causal}\right\},\quad c=?.$$

# Causal optimal transport

- Polish probability spaces $(\mathcal{X}, \mu), (\mathcal{Y}, \nu)$, time horizon $T < \infty$
- Right-continuous filtrations $\mathcal{F}^{\mathcal{X}} = (\mathcal{F}_t^{\mathcal{X}})_{t \in [0,T]}, \mathcal{F}^{\mathcal{Y}} = (\mathcal{F}_t^{\mathcal{Y}})_{t \in [0,T]}$

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**Definition ( Causal transport plans $\Pi^{\mathcal{F}^{\mathcal{X}},\mathcal{F}^{\mathcal{Y}}}(\mu,\nu)$ )**

A transport plan $\pi\in\Pi(\mu,\nu)$ is called causal between $(\mathcal{X},\mathcal{F}^{\mathcal{X}},\mu)$ and $(\mathcal{Y},\mathcal{F}^{\mathcal{Y}},\nu)$ if, for all $t\in[0,T]$ and $D\in\mathcal{F}^{\mathcal{Y}}_t$, the map

$$\mathcal{X}\ni x\mapsto\pi^x(D)$$

is measurable w.t.to $\mathcal{F}^{\mathcal{X}}_t$ ($\pi^x$ regular conditional kernel w.r.t. $\mathcal{X}$).

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**Causal optimal transport problem:**

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# Causal optimal transport

- The concept goes back to Yamada-Watanabe (1971) criterion on solutions of SDEs; see also Jacod (1980), Kurtz (2014), Lassalle (2015), Carmona et al. (2016), Backhoff et al. (2016).

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**Example (Yamada-Watanabe'71)**

Assume weak-existence of the solution to the SDE:

$$dY_t = \sigma(Y_t)dB_t + b(Y_t)dt, \quad b, \sigma \text{ Borel measurable.}$$

Then $(B, Y)_{\#}\mathbb{P}$ is a causal plan between $(\mathcal{C}[0,\infty), \mathcal{F}, B_{\#}\mathbb{P})$ and $(\mathcal{C}[0,\infty), \mathcal{F}, Y_{\#}\mathbb{P})$, where $\mathcal{F}$ is the canonical filtration on $\mathcal{C}[0,\infty)$.

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**From a transport point of view:** from an observed trajectory of $B$, the "mass" can be split at each moment of time into $Y$ only based on the information available up to that time. When there is no splitting of mass (Monge transport), a causal plan is then an actual mapping which is further adapted, i.e. strong solution $Y = F(B)$.

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- Here same filtration. We will instead consider different filtrations (filtration enlargement).

# Causal transport on path space

**Our framework:**

- $\mathcal{X} = \mathcal{Y} = \mathcal{C} := C_0([0,T])$
- $W$ coordinate process on $\mathcal{C}$: $W_t(\omega) = \omega_t$
- $\mathcal{F}^{\mathcal{X}} = \mathcal{F}$ filtration generated by $W$: $\mathcal{F}_t := \bigcap_{u>t} \sigma(W_s, s \le u)$
- $\mathcal{F}^{\mathcal{Y}} = \mathcal{G}$ obtained as enlargement of $\mathcal{F}$ with $G = (g_t(W))_t$:

$$\mathcal{G}_t := \bigcap_{\epsilon>0} \mathcal{G}^0_{t+\epsilon}, \quad \mathcal{G}^0_t := \mathcal{F}_t \vee \sigma(\{G_s, s \le t\}).$$

- given two measures $\mu, \nu$ on $\mathcal{C}$, we will study causal transport plans between $(\mathcal{C}, \mathcal{F}, \mu)$ and $(\mathcal{C}, \mathcal{G}, \nu)$
- we will often consider $\mu = \gamma :=$ Wiener measure on $\mathcal{C}$

# Causal coupling

**Notations:** For a continuous process $Z$ on a $(\Omega, \mathbb{P})$:

$\mathcal{F}^Z := Z^{-1}(\mathcal{F})$ (right-continuous filtration generated by $Z$ on $\Omega$)
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Ex. Initial enlargement: $g_t(Z) = L \ \forall t \geq 0$, $L$ random var. $\mathcal{F}^Z$-mbl
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**Definition (Causal coupling)**

A pair $(X, Y)$ of continuous processes on a probability space $(\Omega, \mathbb{P})$, is called a causal coupling w.r.t. $\mathcal{F}^X$ and $\mathcal{F}^{Y,G}$ if $(X, Y)_{\#}\mathbb{P}$ is a causal transport plan between $(\mathcal{C}, \mathcal{F}, X_{\#}\mathbb{P})$ and $(\mathcal{C}, \mathcal{G}, Y_{\#}\mathbb{P})$.

# Causal coupling

Easy to see, e.g. by Brémaud-Yor (1978):

**Remark ( Characterizations of causality )**

*For a pair $(X, Y)$ of continuous processes on $(\Omega, \mathbb{P})$, TFAE:*

- $(X, Y)$ *is a* causal coupling w.r.t. $\mathcal{F}^X$ *and* $\mathcal{F}^{Y,G}$;
- $\mathbb{P}(D_t \mid \mathcal{F}_t^X) = \mathbb{P}(D_t \mid \mathcal{F}_T^X)$ $\mathbb{P}$-*a.s., for all* $t \in [0, T]$, $D_t \in \mathcal{F}_t^{Y,G}$;
- $\mathcal{F}_t^{Y,G}$ *cond.indep.* $\mathcal{F}_T^X$ *given* $\mathcal{F}_t^X$ w.r.t. $\mathbb{P}$, *for all* $t \in [0, T]$;
- $\mathcal{H}$-*hypothesis holds between* $\mathcal{F}^X$ *and* $\mathcal{F}^X \vee \mathcal{F}^{Y,G}$ w.r.t. $\mathbb{P}$.

*(every sq.integrable* $\mathcal{F}^X$*-mart. is a sq.integrable* $\mathcal{F}^X \vee \mathcal{F}^{Y,G}$*-mart.)*

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**Mass transport interpretation:** At every time the mass transported to the $2^{nd}$ process is only based on the information on the $1^{st}$ process up to that time (+ something independent of the whole $1^{st}$ process).

# Causal coupling: Brownian case

**Lemma**

*Let $X$ be a Brownian motion and $Y$ a continuous process on $(\Omega, \mathbb{P})$. Then $(X, Y)$ is a causal coupling w.r.t. $\mathcal{F}^X$ and $\mathcal{F}^{Y,G}$ IFF $X$ is a Brownian motion in $\mathcal{F}^X \vee \mathcal{F}^{Y,G}$.*

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Motivating example to study causal coupling in a filtration enlargement framework:

**Example**

Let $B$ be a Brownian motion on $(\Omega, \mathbb{P})$, which remains a semi-martingale w.r.t. the enlarged filtration $\mathcal{F}^{B,G}$, with decomposition

$$dB_t = d\tilde{B}_t + dA_t.$$

Then, for any $T > 0$, $(\tilde{B}, B)$ is a causal coupling w.r.t. $\mathcal{F}^{\tilde{B}}$ and $\mathcal{F}^{B,G}$, that is, $(\tilde{B}, B)_{\#}\mathbb{P} \in \Pi^{\mathcal{F},\mathcal{G}}(\gamma, \gamma)$.

# Value of additional information

- **Aim:** use causal transport framework to give an estimate of the value of the additional information, for some classical stochastic optimization problems (difference of optimal value of these problems with or without additional information).
- **Idea:** take projection w.r.t. causal couplings of the optimizers in the problem with the larger filtration (additional information), so building a feasible element in the problem with the smaller filtration and making a comparison possible.
- Pflug (2009) uses this idea in discrete-time, to gauge the dependence of multistage stochastic programming problems w.r.t. different reference probability measures.

# Utility maximisation

- $B$ $d$-dimensional Brownian motion on $(\Omega, \mathbb{P})$.
- Financial market: riskless asset $\equiv 1$, and $m \leq d$ risky assets:

$$dS_t^i = S_t^i \Big( b_t^i \, dt + \sum_{j=1}^{d} \sigma_t^{ij} dB_t^j \Big), \qquad i = 1, ..., m.$$

- $|b_t^i(\omega) - b_t^i(\tilde{\omega})| \leq L \sum_{k=1}^{d} \sup_{s \leq t} |\omega_s^k - \tilde{\omega}_s^k|$, same for $\sigma^{ij}$, $\sigma$ bdd

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- $\lambda_t^i$: proportion of an agent's wealth invested in the $i^{th}$ stock at time $t$: assume $\lambda_t^i \in [0, 1]$ (**no short-selling**)
- $\mathcal{A}(\mathcal{F}^B)$: set of admissible portfolios for the agent without anticipative information ($\mathcal{F}^B$-progressively measurable $\lambda$)
- $\mathcal{A}(\mathcal{F}^{B,G})$: set of admissible portfolios for the agent wit anticipative information ($\mathcal{F}^{B,G}$-progressively measurable $\lambda$)

# Utility maximisation

→ We want to compare the utility maximization problems:

$$v = \sup_{\lambda \in \mathcal{A}(\mathcal{F}^B)} \mathbb{E}[U(X_T^\lambda)], \qquad v(G) = \sup_{\lambda \in \mathcal{A}(\mathcal{F}^{B,G})} \mathbb{E}[U(X_T^\lambda)].$$

- $(X_t^\lambda)_t$: wealth process corresponding to $\lambda$, $X_0^\lambda = 1$.
- utility function $U : \mathbb{R}_+ \to \mathbb{R}$ concave, increasing, and s.t.

$$F := U \circ exp \quad \text{is C-Lipschitz, concave and increasing.}$$

e.g. $U(x) = \frac{x^a}{a}$, $a \leq 0$; $U(x) = ln(x)$; $U(x) = -\frac{1}{a}e^{-ax}$, $a \geq 1$

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**Proposition**

*The following bound holds, for a specific constant $K$:*

$$0 \leq v(G) - v \leq K \inf_{\pi \in \Pi^{\mathcal{F},\mathcal{G}}(\gamma,\gamma)} \mathbb{E}^\pi[V_T(\bar{\omega} - \omega)].$$

$(\omega, \bar{\omega})$: generic element in $\mathcal{C} \times \mathcal{C}$, $V_T$: total variation up to time $T$.

# Utility maximisation

**Remark.** In a complete market, for log utility, and for initial enlargements of filtrations, the difference $v(G) - v$ is known explicitly (Pikovsky-Karatzas 1996).

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- $(\pi, \mathcal{F} \times \{\emptyset, C\})$-optional projection: $\tilde{\lambda} \in \mathcal{A}(\mathcal{F}^B)$
- in particular $\tilde{\lambda}_t(\omega) = \tilde{\lambda}_t(\omega, \bar{\omega}) = \mathbb{E}^{\pi}[\hat{\lambda}_t | \mathcal{F}_t] = \mathbb{E}^{\pi}[\hat{\lambda}_t | \mathcal{F}_T]$
- substitute in $v$

# Optimal stopping

- With the same method used above, we can estimate the value of information wrt other optimization problems, e.g.

$$v := \inf_{\mathcal{F}^W\text{-st.t.}} \mathbb{E}^{\mathbb{P}}\left[\ell(W,\tau)\right], \quad v(G) := \inf_{\mathcal{F}^{W,G}\text{-st.t.}} \mathbb{E}^{\mathbb{P}}\left[\ell(W,\tau)\right],$$

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**Proposition**

*Let $\ell$ be $K$-Lipschitz in its first argument wrt a metric $d$ on $\mathcal{C}\times\mathcal{C}$, uniformly in time. Then*
$$0 \le v - v(G) \le K \inf_{\pi\in\Pi^{\mathcal{F},\mathcal{G}}(\gamma,\gamma)} \mathbb{E}^{\pi}[d(\omega,\bar\omega)].$$

E.g. $\ell(x,t) = f(x_t)$ and $\ell(x,t) = f(\sup_{s\le t} x_s)$ satisfy the above conditions, with $d(\omega,\tilde\omega) = \|\omega-\tilde\omega\|_\infty$, if $f$ is Lipschitz. In this case
$$0 \le v - v(G) \le K \inf_{\pi\in\Pi^{\mathcal{F},\mathcal{G}}(\gamma,\gamma)} \mathbb{E}^{\pi}[V_T(\bar\omega-\omega)].$$

# Concluding remarks

- We impose the causal constraint on transport plans $\rightarrow$ causal optimal transport problem (classical attainability and duality results can be shown for such problems).
- With **cost function = total variation**, the causal optimal transport problem can be used to estimate the value of additional information in several optimization problems.
- With the **same cost function**, the causal optimal transport problem can be used to characterize the preservation of semimartingale property in enlarged filtrations.

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