Branching diffusion representation for semilinear PDEs and Monte-Carlo approximation

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Outline

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Feynmann-Kac formula

- Let $W_\cdot$ be a standard $d-$dimensional Brownian motion, $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ be continuous and of polynomial growth.
  - The heat equation on $[0,T] \times \mathbb{R}^d$:

$$\Big(\partial_t u + \frac{1}{2}\Delta u + f\Big)(t,x) = 0, \quad u(T,\cdot) = g(\cdot). \tag{1}$$

  - A probabilistic representation

$$u(t,x) \;=\; \mathbb{E}\Big[g(x + W_{T-t}) + \int_t^T f(s, x + W_{s-t})ds\Big]. \tag{2}$$

Theorem (Feynmann-Kac et the reverse)

(i) *Let $u \in C^{1,2}([0,T] \times \mathbb{R}^d)$ be a classical solution of PDE* (1), *then u has the probabilistic representation* (2).
(ii) *Let the function $u(t,x)$ be defined by* (2), *then u is a "solution" of PDE* (1).

BSDE and semi-linear PDE (Pardoux-Peng)

- The semillinear parabolic PDE :

$$\partial_t u(t,x) + \frac{1}{2}\Delta u(t,x) + f(\cdot, u, Du)(t,x) = 0, \qquad u(T,\cdot) = g(\cdot).$$

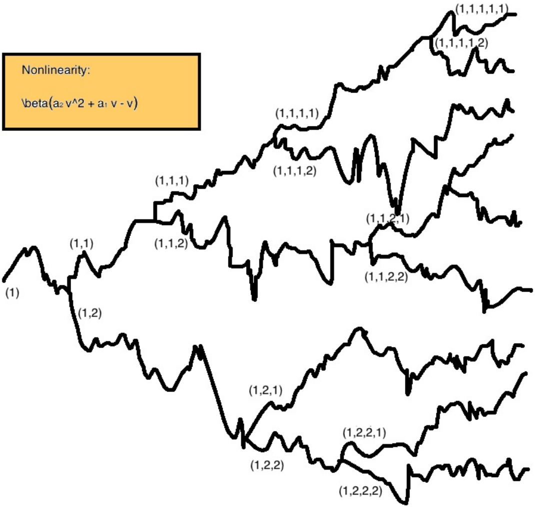
- The backward SDE :

$$Y_t \;=\; g(W_T) + \int_t^T f(s, W_s, Y_s, Z_s)ds + \int_t^T Z_s dW_s.$$

Theorem

(i) Let $u \in C^{1,2}([0,T] \times \mathbb{R}^d)$ be a solution of the semilinear PDE, then $(Y_t, Z_t) := (u(t,W_t), Du(t,W_t))$ provides a solution of the the BSDE,
(ii) Define $u(t,x) := Y_t^{t,x}$, then u is a "solution" of semilinear PDE.

Branching diffusion process



Branching diffusion process and semi-linear PDE

$$\mathcal{K}_t := \{\text{All particles alive at time } t\}, \quad \overline{\mathcal{K}}_T := \cup_{t \le T} \mathcal{K}_t.$$

• [Skorokhod, Watanebe, McKean, etc.] Representation of KPP equation

$$\partial_t u + \frac{1}{2}\Delta u + \beta\Big(\sum_{\ell \in \mathbb{N}} p_\ell u^\ell - u\Big) = 0, \quad u(T, \cdot) = g(\cdot),$$

by branching Brownian motion $\mathbb{E}\Big[\prod_{k \in \mathcal{K}_T} g(W_T^k)\Big]$.

• [Henry-Labordère, Henry-Labordère-Tan-Touzi] Extension and Monte-Carlo method

$$\partial_t u + \frac{1}{2}\Delta u + \beta\Big(\sum_{\ell \in \mathbb{N}} p_\ell a_\ell u^\ell - u\Big) = 0, \quad u(T, \cdot) = g(\cdot),$$

by
$$\mathbb{E}\Big[\Big(\prod_{k \in \overline{\mathcal{K}}_T \setminus \mathcal{K}_T} a_{I_k}(T_k, W_{T_k}^k)\Big)\Big(\prod_{k \in \mathcal{K}_T} g(W_T^k)\Big)\Big].$$

Objective

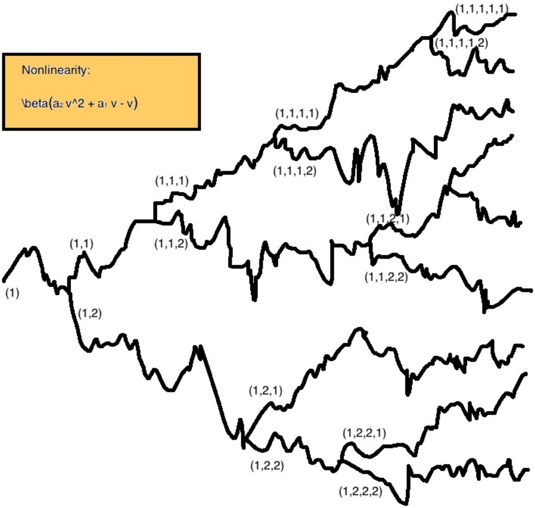
- Extension

$$\sum_{\ell \in \mathbb{N}} p_\ell a_\ell u^\ell \quad \longrightarrow \quad \sum_{\ell=(\ell_0,\ell_1,\cdots,\ell_m)} p_\ell a_\ell \; u^{\ell_0} \; \Pi_{i=1}^m (b_i \cdot Du)^{\ell_i}.$$

- Forward numerical algorithm for semilinear PDE.

Outline

A branching Brownian motion



A branching Brownian motion

A branching Brownian motion $(W^k)_{k \in \overline{\mathcal{K}}_T}$:

- $\mathcal{K}_t$ the collection of particles alive at time t, $\overline{\mathcal{K}}_T := \cup_{t \le T} \mathcal{K}_t$.
- $\mathcal{K}_t^n \subset \mathcal{K}_t$ (resp. $\overline{\mathcal{K}}_T^n$) the collection of particles of generation n.
- A particle $k \in \overline{\mathcal{K}}_T$ defaults at time $T_k \wedge T$.
- For a particle $k \in \overline{\mathcal{K}}_T$, denote by $k-$ its parent particle, then the particle starts its life at time T_{k-} and ends its life at time $T_k \wedge T$.
- $\Delta T_k := T_k - T_{k-}$ follows a distribution of density ρ (e.g. $\mathcal{E}(\beta)$ with $\rho(t) = \beta e^{-\beta t} \mathbb{1}_{t \ge 0}$), denote $\overline{F}(T) := \int_T^\infty \rho(s) ds$.
- At default time $T_k < T$, the particle k branches into I_k offspring particles, where $\mathbb{P}(I_k = \ell) = p_\ell$.
- For $k \in \overline{\mathcal{K}}_T$, $W^k_{T_{k-}} = W^{k-}_{T_{k-}}$.

A KPP equation and branching Brownian motion

- KPP equation : Let $u \in C^{1,2}$ be a solution to

$$\partial_t u + \frac{1}{2}\Delta u + \sum_{\ell \in \mathbb{N}} a_\ell u^\ell = 0, \quad u(T,\cdot) = g(\cdot).$$

- Using Feynmann-Kac

$$\begin{aligned} u(0,0) &= \mathbb{E}\Big[g(W_T) + \int_0^T \sum_\ell (a_\ell u^\ell)(s, W_s) ds\Big] \\ &= \mathbb{E}\Big[\frac{g(W_T)}{\overline{F}(T)}\overline{F}(T) + \int_0^T \sum_\ell (a_\ell u^\ell)(s, W_s)\frac{1}{\rho(s)}\rho(s) ds\Big] \\ &= \mathbb{E}\Big[\frac{g(W_T^{(1)})}{\overline{F}(T)} \mathbb{1}_{\{T_{(1)} \geq T\}} + \frac{a_{I_{(1)}}(T_{(1)}, W^{(1)}_{T_{(1)}})}{p_{I_{(1)}}\rho(T_{(1)})} u^{I_{(1)}}_{T_{(1)}} \mathbb{1}_{\{T_{(1)} < T\}}\Big]. \end{aligned}$$

A KPP equation and branching Brownian motion ...

- Let us introduce

$$\psi_1 := \Big[\prod_{k\in\mathcal{K}_T^1} \frac{g(W_T^k)}{\overline{F}(\Delta T_k)}\Big]\Big[\prod_{k\in\overline{\mathcal{K}}_T^1\setminus\mathcal{K}_T^1} \frac{a_{I_k}(T_k, W_{T_k}^k)}{p_{I_k}\rho(T_k)}\Big]\Big[\prod_{k\in\overline{\mathcal{K}}_T^2} u_{T_{k-}}\Big].$$

Then

$$\begin{aligned} u(0,0) &= \mathbb{E}\Big[\frac{g(W_T^{(1)})}{\overline{F}(T)}\mathbb{1}_{\{T_{(1)}\geq T\}} + \frac{a_{I_{(1)}}(T_{(1)}, W_{T_{(1)}}^{(1)})}{p_{I_{(1)}}\rho(T_{(1)})} u_{T_{(1)}}^{I_{(1)}} \mathbb{1}_{\{T_{(1)}<T\}}\Big] \\ &= \mathbb{E}\big[\psi_1\big]. \end{aligned}$$

A KPP equation and branching Brownian motion ...

- Recall

$$\psi_1 := \Big[\prod_{k\in\mathcal{K}_T^1} \frac{g(W_T^k)}{\overline{F}(\Delta T_k)}\Big]\Big[\prod_{k\in\overline{\mathcal{K}}_T^1\setminus\mathcal{K}_T^1} \frac{a_{I_k}(T_k, W_{T_k}^k)}{p_{I_k}\rho(T_k)}\Big]\Big[\prod_{k\in\overline{\mathcal{K}}_T^2} u_{T_{k-}}\Big]$$

and $$u(0,0) = \mathbb{E}\big[\psi_1\big].$$

- By iteration, one has $u(0,0) = \mathbb{E}\big[\psi_n\big] = \mathbb{E}\big[\lim_{n\to\infty}\psi_n\big] = \mathbb{E}\big[\psi\big]$, where

$$\psi_n := \Big[\prod_{k\in\cup_{j=1}^n\mathcal{K}_T^j} \frac{g(W_T^k)}{\overline{F}(\Delta T_k)}\Big]\Big[\prod_{k\in\cup_{j=1}^n\overline{\mathcal{K}}_T^j\setminus\mathcal{K}_T^j} \frac{a_{I_k}(T_k, W_{T_k}^k)}{p_{I_k}\rho(T_k)}\Big]\Big[\prod_{k\in\overline{\mathcal{K}}_T^{n+1}} u_{T_{k-}}\Big],$$

$$\psi := \Big[\prod_{k\in\mathcal{K}_T} \frac{g(W_T^k)}{\overline{F}(\Delta T_k)}\Big]\Big[\prod_{k\in\overline{\mathcal{K}}_T\setminus\mathcal{K}_T} \frac{a_{I_k}(T_k, W_{T_k}^k)}{p_{I_k}\rho(T_k)}\Big].$$

A second class of seminlinear equation

- A second class of semilinear PDEs : Let $u \in C^{1,2}$ be a solution to

$$\partial_t u + \frac{1}{2}\Delta u + \sum_{\ell=(\ell_0,\ell_1)} a_\ell u^{\ell_0}(b \cdot Du)^{\ell_1} = 0, \quad u(T,\cdot) = g(\cdot).$$

- Using Feynmann-Kac (let $\mathbb{P}[I_{(1)} = \ell = (\ell_0, \ell_1)] = p_\ell$)

$$u(0,0) = \mathbb{E}\Big[g(W_T) + \int_0^T \sum_\ell \big(a_\ell u^{\ell_0}(b \cdot Du)^{\ell_1}\big)(s, W_s)ds\Big]$$

$$= \mathbb{E}\Big[\frac{g(W_T^{(1)})}{\overline{F}(T)} \mathbb{1}_{\{T_{(1)} \geq T\}} + \frac{a_{I_{(1)}}(T_{(1)}, W)}{p_{I_{(1)}}\rho(T_{(1)})} u^{I_{(1),0}}(b \cdot Du)^{I_{(1),1}}_{T_{(1)}} \mathbb{1}_{\{T_{(1)} < T\}}\Big]$$

$$= \mathbb{E}[\psi_1],$$

where $\psi_1 := \Big[\prod_{k \in \mathcal{K}_T^1} \frac{g(W_T^k)}{\overline{F}(\Delta T_k)}\Big]\Big[\prod_{k \in \overline{\mathcal{K}}_T^1 \setminus \mathcal{K}_T^1} \frac{a_{I_k}(T_k, W_{T_k}^k)}{p_{I_k}\rho(T_k)}\Big]\Big[\prod_{k \in \overline{\mathcal{K}}_T^2} u_{T_{k-}} \text{ or } (b \cdot Du)\Big]$.

Seminlinear equation and branching Brownian motion

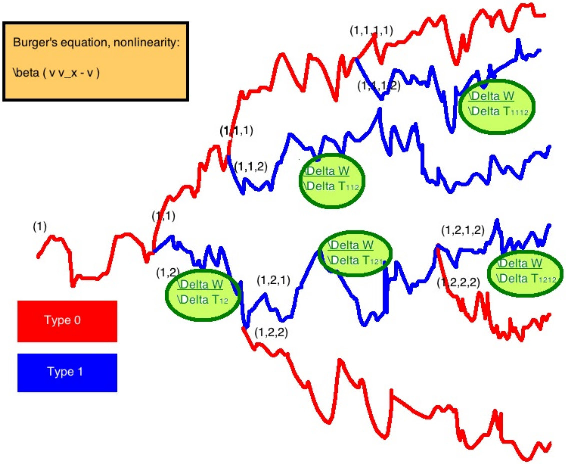
- Modification on Branching process :
  - $\mathbb{P}[I_k = \ell = (\ell_0, \ell_1)] = p_\ell$,
  - At default time, the particle k branches into $|I_k|$ (independent) particles, among which $I_{k,0}$ particles are marked by 0, and $I_{k,1}$ particles are marked by 1.
  - Denote by θ_k the mark of k (initial particle is marked by 0).
- Automatic differentiation, let $v(x) = \mathbb{E}[\phi(x + \Delta W)]$, then by integral by part formula,

$$Dv(x) \;=\; \mathbb{E}\Big[\phi(x + \Delta W)\frac{\Delta W}{\Delta T}\Big]$$

- Then

$$(b \cdot Du)(0,0) \;=\; \mathbb{E}\Big[\psi_1 \; b(0,0) \cdot \frac{\Delta W_{(1)}}{\Delta T_{(1)}}\Big].$$

A branching Brownian motion



Seminlinear equation and branching Brownian motion

- For every $k \in \overline{\mathcal{K}}_T$, introduce

$$\mathcal{W}_k \ := \ \mathbb{1}_{\{\theta_k=0\}} + b(T_{k-}, W_{T_{k-}}) \cdot \frac{\Delta W_k}{\Delta T_k} \mathbb{1}_{\{\theta_k \neq 0\}}.$$

Then $u(0,0) = \mathbb{E}[\psi_n] = \mathbb{E}[\psi]$, where

$$\psi_n \ := \ \Big[\prod_{k \in \cup_{j=1}^n \mathcal{K}_T^j} \frac{g(W_T^k) \mathcal{W}_k}{\overline{F}(\Delta T_k)} \Big] \Big[\prod_{k \in \cup_{j=1}^n \overline{\mathcal{K}}_T^j \setminus \mathcal{K}_T^j} \frac{a_{I_k}(T_k, W_{T_k}^k) \mathcal{W}_k}{p_{I_k} \rho(T_k)} \Big] \Big[\prod_{k \in \overline{\mathcal{K}}_T^{n+1}} \big(u_{T_{k-}} \mathbb{1}_{\{\theta_k=0\}} + (b \cdot Du)_{T_{k-}} \mathbb{1}_{\{\theta_k=1\}} \Big],$$

and

$$\psi := \Big[\prod_{k \in \mathcal{K}_T} \frac{g(W_T^k) \mathcal{W}_k}{\overline{F}(\Delta T_k)} \Big] \Big[\prod_{k \in \overline{\mathcal{K}}_T \setminus \mathcal{K}_T} \frac{a_{I_k}(T_k, W_{T_k}^k) \mathcal{W}_k}{p_{I_k} \rho(T_k)} \Big].$$

Drawback and extensions (Bouchard-Tan-Zou)

• It only works for small non-linearity $|a_\ell| \ll 1$ or short maturity $T \ll 1$. But approximation of a Lipschitz function $f(u, Du)$ may lead to big non-linearity.

• Locally polynomial approximation :

$$f(u) \;\approx\; f_\circ(u,u) = \sum_{\ell=0}^{\ell_\circ} a_\ell(u) u^\ell.$$

• A Picard iteration scheme

$$\partial_t u^{n+1} + \frac{1}{2}\Delta u^{n+1} + f_\circ(u^n, u^{n+1}) = 0.$$

Drawback and extensions

• One can formally deduce an estimator for fully nonlinear case, where the nonlinearity is given by

$$f(u, Du, D^2u) := \sum_{\ell_0,\ell_1,\ell_2} a_\ell u^{\ell_0}(Du)^{\ell_1}(D^2u)^{\ell_2}.$$

But the estimator is not integrable because of the Malliavin weight term :

$$\mathbb{E}\big[D^2\varphi(x+W_T)\big] \;=\; \mathbb{E}\Big[\varphi(x+W_T)\frac{W_T^2-T}{T^2}\Big].$$

• See Xavier's talk ...