Trading with Market Impact

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I have benefited the collaboration of

Albert Altarovici, Peter Bank, Bruno Bouchard, Umut Çetin,
Yan Dolinsky, Selim Gökay, Ludovic Moreau,
Johannes Muhle-Karbe, Dylan Possamaï, Max Reppen,
Nizar Touzi, Moritz Voss.

Consider a situation in which there is a target portfolio. However due to frictions it is not possible to track it perfectly. Then, the natural question is devise a tracking mechanism which is optimal in the given the structure.

So we need to model :

- the target portfolio : this can be given exogenously or obtained as a solution of a frictionless optimisation problem.
- the financial frictions,
- optimisation criterion.

In earlier works, this portfolio is taken as the solution to a Merton type utility maximization problem in a frictionless market. Then, the tracking portfolio is constructed as the solution of the same utility maximization problem in the actual market. When the underlying Merton problem is naturally given, this approach has many advantages. In particular, the target portfolio has structure and the optimisation criterion is clear.

However, there are many situations in which the target is given exogenously. With Bank & Voss, we formulate this problem with a given stochastic process.

- Frei & Westray *Math. Finance* (2015) and Cartea & Jaimungal SIFIN, (2016) consider solve a Markovian optimal liquidation. Techniques are dynammic programming type.
- Naujokat & Westray *MAFE* (2011) "*Curve following in illiquid markets*" use the BSDE theory to characterize the solution.
- Kohlmann & Tang (2002) studies a stochastic linear quadratic regulator problem and derives Ricatti type BSDE.
- Cai, Rosenbaum & Tankov preprints (2016) start with a given diffusion process. They consider a general cost structure and obtain tracking portfolios which are asymptotically optimal.

We consider a general target, and obtain an explicit solution in the quadratic set-up.

Natural markets frictions are

- Transaction costs due to bid-ask spread. This is the proportional transaction costs ;
- Fixed transaction costs ;
- Trading costs due to market impact ;
- Discrete time effects.

This is a completely context depended choice. Several examples are

- Expected utility when target is given through such a problem ;
- A function of hedging costs when the target is the hedging portfolio for a derivative ;
- A general criteria depending on the deviation from the target i.e., tracking error and the frictional costs i.e., tracking effort.

Fix an underlying probabilistic structure $(\Omega, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$.

Assume that we are given a predictable target portfolio, $\{\xi_t\}_{t\in[0,T]}$.

Given an initial portfolio x, the problem is to minimize

$$\psi \in \mathcal{A}_x \quad \rightarrow \quad J(\psi) := \mathbb{E}\int_0^T \left[|\xi_t - \psi_t|^2 + \kappa|\dot{\psi}_t|^2\right] \, dt,$$

where $\kappa > 0$ is a given gearing factor (related to utility if there is one), the admissible class $\mathcal{A}_x$ is simply all differentiable and adapted processes starting at x.

There are two terms in the cost functional :

Tracking Error : $\mathbb{E} \int_0^T |\xi_t - \psi_t|^2 \; dt,$

Tracking Effort : $\mathbb{E} \int_0^T |\dot{\psi}_t|^2 \; dt.$

The second one is related to market impact models in which the price impact of the trade ψ is modelled through $|\dot{\psi}_t|^2$ as in Almgren & Chriss.

The choice for quadratic structure will be motivated later. But it also makes the problem tractable.

This is a linear quadratic regulator type problem. However, ξ is given not through a differential equation but rather as a process.

Roger & Singh (2007) considers a similar problem in the Black Scholes model. The optmisation problem they consider is to minimise

$$J^{\epsilon}(\psi) := \mathbb{E}\int_0^T \left[(\psi_t - \theta(t,S_t))\sigma^2 S_t^2 + \epsilon S_t(\dot{\psi}_t)^2\right] \, dt,$$

where $dS_t = S_t(\mu dt + \sigma dW_t)$ and θ is the hedging portfolio of a derivative in the complete Black Scholes market.

In our context, the target is $\xi = \theta(t, S_t)$ and the optimisation problem is slightly different.

They propose an approximate solution to this problem using the dynamic programming.

Theorem (Bank, Soner, Voss, MAFE 2016)

The optimal solution ψ^* is given as the solution of

$$\dot{\psi}^*_t = C(t,\kappa)\ (\hat{\xi}_t - \psi^*_t), \qquad \psi^*_0 = x,$$

where $C(t,\kappa) = \tanh(\tau(t,\kappa))/\sqrt{\kappa}$, $\tau(t,\kappa) := (T-t)/\sqrt{\kappa}$,

$$\hat{\xi}_t := \mathbb{E}\left[\int_t^T K(t,u)\ \xi_u\ du \mid \mathcal{F}_t\ \right],$$

and the kernel $K(t,\cdot)$ is given by

$$K(t,u) := \frac{\cosh(\tau(u,\kappa))}{\sqrt{\kappa}\ \sinh(\tau(t,\kappa))} \ \Rightarrow\ K(t,u) > 0 \text{ and } \int_t^T K(t,u)du = 1.$$

The optimal solution targets not ξ but rather $\hat{\xi}$ given by

$$\hat{\xi}_t := \mathbb{E}\left[\int_t^T K(t,u)\ \xi_u\ du \mid \mathcal{F}_t\right].$$

The modified target is the conditional expectation of the weighted future average of the original ξ.

The Kernel $K(t,\cdot)$ in fact depends on κ as well. The smaller the κ is the more concentrated it is around t. In fact,

$$K(t,u)du \quad \to \quad \delta_t(du) \qquad \text{as } \kappa \text{ tends to } 0.$$

Garleanu & Pedersen quote Wayne Gretzky, "*A great hockey player skates to where the puck is going to be, not where it is.*"

The minimal value is also explicitly avaliable :

$$\min_{\psi} J(\psi) = c\sqrt{\kappa}(x-\hat{\xi}_0)^2+\mathbb{E}\int_0^T(\xi_t-\hat{\xi}_t)^2dt+\sqrt{\kappa}\,\mathbb{E}\int_0^T c(t)d\langle\,\hat{\xi}\,\rangle_t.$$

We always assume that ξ is square integrable, so the second term is always finite.

However, the modified target $\hat{\xi}$ and its quad If ξ has quadratic variation then the second quadratic variation depend on κ. In particular, if ξ does not have finite quadratic variation, the third term may not be order $\sqrt{\kappa}$.

- Çetin-Jarrow-Protter model of liquidity is the representative for this type of models. In this setting the authors postulate the existence of a supply curve for the price process of the asset.
- The supply curve gives you the price per share once you specify the time and size of the trade.
- All investors are price takers to the supply curve and have no lasting impact on the evolution of the underlying.

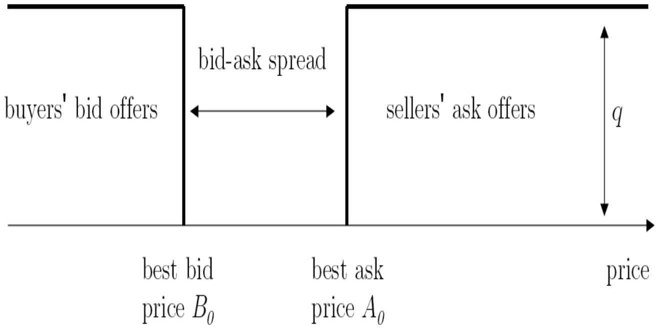


Fig. 1. The limit order book model before the large investor is active

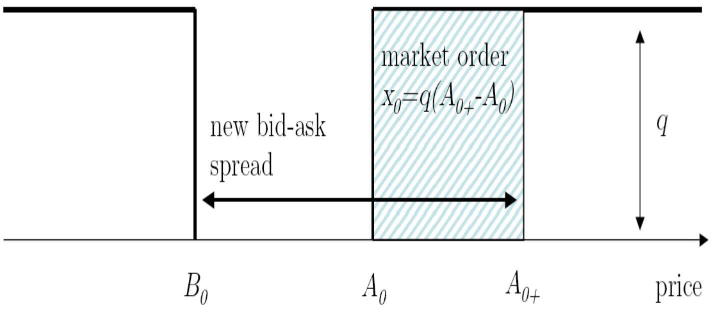


Fig. 2. Impact of a market buy order of x_0 shares

This is a phenomenological model by Almgren & Chriss (also important contributions by Rogers & Sign, Garleanu & Pedersen), considers an impact functional of the form

$$\mathcal{S}(t, S_t, \dot{\psi}_t) = S_t + \Lambda_t \dot{\psi}_t,$$

where ψ_t is the portfolio position. Then, the wealth dynamics are given by

$$Y_t^{\psi} = \int_0^t \psi_u dS_u - L_t$$

$$L_t = \int_0^t \Lambda_u \left(\dot{\psi}_u\right)^2 du.$$

In these models, it is not possible to avoid the liquidity premium.

Consider a utility maximization problem

$$\boxed{\sup_{\psi} \mathbb{E}\left[\, U\left(\mathcal{R}_T^{\psi}\right)\,\right],}$$

where $\mathcal{R}_T^{\psi}$ is the risk adjusted liquidation cost of Schöneborn and is given by,

$$\mathcal{R}_T^{\psi} := Y_T^{\psi} - C\Lambda^2(\psi_T - \psi_T^*)^2,$$

where C is a constant derived from the model and ψ^* is optimal portfolio for the frictionless (i.e., $\Lambda = 0$) market.

There are several difficulties :

- Due to the price impact, we could only use portfolios that are differentiable in time.
  On the other hand the target portfolio ψ^* is the the optimal strategy for the frictionless problem and almost always rough.
- One can consider the optimal solution ψ^Λ of the problem with frictions, as an optimal tracking of ψ^*.
- In addition to continuous targeting error, we have both initial and final liquidation costs. Initially, we might be far from the optimal location and need to move there efficiently.
- Closer to maturity one must consider the final portfolio position.

The actual problem is not tractable and together with Moreau & Muhle-Karbe we considered the asymptotics as Λ gets smaller.

We have asymptotic results for the value function and also for optimal portfolio.

The rigorous proof uses machinery from viscosity solutions developed by Possamaï, Soner & Touzi. I do not report it here. Only I will briefly outline the asymptotic structure of the hedge.

Let $\psi^* = \psi^{*,\Lambda}$ be the optimal portfolio for the utility maximization problem with small but non-zero impact $\Lambda > 0$. And let ξ be the frictionless optimizer.

Asymptotically,

$$\frac{d}{dt}\psi^*_t = \frac{c}{\sqrt{\Lambda}}\,(\xi_t - \psi^*_t), \quad \text{where } c = \frac{\sigma}{\sqrt{2R_t}},$$

and R_t is the frictionless investor's indirect risk-tolerance process, i.e., the risk tolerance of the frictionless value.

As Λ gets smaller, ψ^* moves very quickly towards the frictionless optimizer ξ.

The asymptotic solution for the utility maximization problem is of the form

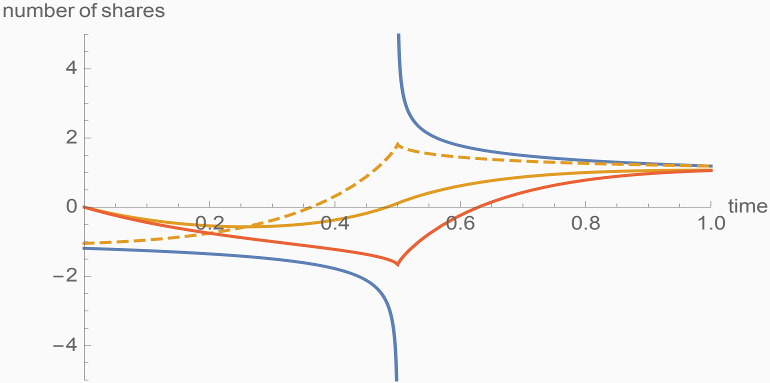
$$\frac{d}{dt}\psi_t^* = \frac{c}{\sqrt{\Lambda}}\,(\xi_t - \psi_t^*).$$

While the solution of the tracking problem has the form

$$\frac{d}{dt}\psi_t^* = \frac{C(t,\kappa)}{\sqrt{\kappa}}\,(\hat{\xi}_t - \psi_t^*).$$

So Λ plays the role of κ.

Then, the main difference is that in the tracking problem one aims at the weighted avarege of the future estimate, $\hat{\xi}$.



Here ξ=blue, $\hat{\xi}$= dashed orange, ψ^*= orange. The red curve is the solution to myopic one, i.e, aiming at ξ and not $\hat{\xi}$.

- In the diffusion context of Moreau, Muhle-Karbe & Soner the difference between frictionless value function and the frictional one is shown to be $\sqrt{\Lambda}$. (Recall that κ and Λ play the same role). However, in our tracking problem scaling may not be order $\sqrt{\kappa}$ when ξ is rough.
- So this might be the reason behind some of the smoothness assumptions in Bichuch and Bouchard, Moreau & Soner. These papers prove the so-called Whalley- Wilmott asymptotics for the Davis price of a derivative with proportional transaction costs. Near maturity, the Black-Scholes hedge may get rough unless the pay-off is smooth. And this might cause the tracking error to scale differently with respect to the small parameter.

- A related model is the friction due to transaction costs. Asymptotics analysis has been successfully used in that context by Shreve and collaborators, by myself with Altarovici, Reppen, Muhle-Karbe, Touzi.
- Relatedly, in a series of papers Kallsen & Muhle-Karbe studied directly the asymptotics of the optimal portfolio.
- Kallsen & Muhle-Karbe formulae although different in their fine details, have many common features. In particular, the risk tolerance function plays a central role. But the scaling in the small parameter might be different.

Outline

In this section, I outline

- The proof for the tracking problem ;
- A formal connection to the utility maximization problem.

$$J(\psi) = \frac{1}{2}\mathbb{E}\int_0^T \left[|\xi_t - \psi_t|^2 + \kappa|\dot{\psi}_t|^2\right] \, dt.$$

Consider the Gateux derivative in the direction of φ :

$$\begin{aligned}\langle J'(\psi^*), \varphi\rangle &= \mathbb{E}\int_0^T \left[(\xi_t - \psi_t^*)\varphi_t + \kappa\dot{\psi}_t^* \ \dot{\varphi}_t\right] \, dt \\ &= \mathbb{E}\int_0^T \left[(\Xi_t + \kappa\dot{\psi}_t^*) \ \dot{\varphi}_t\right] \, dt,\end{aligned}$$

where

$$\Xi_t := \int_t^T (\psi_u^* - \xi_u) \, du.$$

Since $\Xi_t := \int_t^T (\psi_u^* - \xi_u)\, du$ and

$$0 = \langle J'(\psi^*), \varphi \rangle = \mathbb{E} \int_0^T \left[\left(\Xi_t + \kappa \dot{\psi}_t^* \right) \dot{\varphi}_t \right] \, dt,$$

for every perturbation φ, $\kappa \dot{\psi}_t^* + \mathbb{E}[\ \Xi_t \mid \mathcal{F}_t\ ] = 0$. Hence,

$$d\dot{\psi}_t^* = \frac{1}{\kappa} [(\psi_t^* - \xi_t) dt + dM_t],$$

$$M_t = \mathbb{E} \left[\int_0^T (\psi_u^* - \xi_u) du \mid \mathcal{F}_t \right].$$

We now show that the solution with $\dot{\psi}_T^* = 0$ is given as announced.

Assume first ξ_t is deterministic.

We make an ansatz that $\dot{\psi}_t = c(t)(z(t) - \psi_t)$. Then, derive formally the equations,

$$\dot{c}(t) = \frac{c(t)^2}{\kappa} - 1, \qquad \dot{z}(t) = \frac{c(t)z(t)}{\kappa} - \xi_t.$$

We can solve these equations.

In the stochastic context, either one uses the stochastic Riccati equations (see recent work of Bank & Voss) or simply formally take the conditional expectation. Then, verify that the suggested solution indeed solves the first order conditions.

Outline

For $\Lambda \geq 0$ and a utility function U, set

$$V^{\Lambda} := \sup_{\psi} J^{\Lambda}(\psi) := \mathbb{E}\left[U(Y_T^{\Lambda,\psi} - \eta)\right],$$

where the wealth process $Y^{\Lambda,\psi}$ with impact is given by,

$$Y_T^{\Lambda,\psi} = y + \int_0^T \psi_t dS_t - \Lambda \int_0^T (\dot{\psi}_t)^2 dt$$

and the random endowment η has the form

$$\eta = \int_0^T \xi_t dS_t.$$

For $\Lambda = 0$ $\psi \equiv \xi$ is optimal. Then, $V^0 = U(0) = 0$. Taylor expansion implies that

$$\begin{aligned} J^\Lambda(\psi) \approx & \mathbb{E}\left[U'(0) Y_T^{\Lambda,\psi} + \frac{1}{2} U''(0) (Y_T^{\Lambda,\psi})^2\right] \\ = & \mathbb{E}\left[-U'(0)\ \Lambda \int_0^T (\dot{\psi}_t)^2 dt \right. \\ & \left. + \frac{1}{2} U''(0) \int_0^T (\psi_t - \xi_t)^2 d\langle S \rangle_t\right]. \end{aligned}$$

So original maximization problem is approximately same as the original tracking problem with

$$\kappa = \Lambda\ \frac{-U'(0)}{U''(0)}.$$

- There are a rich class of models for illiquid markets with price impact.
- Another use of this approach is to assume that the target portfolio is given but not implementable. This would give us away to provide implementable approximations.
- Asymptotics makes things tractable.

THANK YOU FOR YOUR ATTENTION.