# Markovian and product quantization of an $\mathbb{R}^d$-valued Euler scheme of a diffusion process with applications to finance

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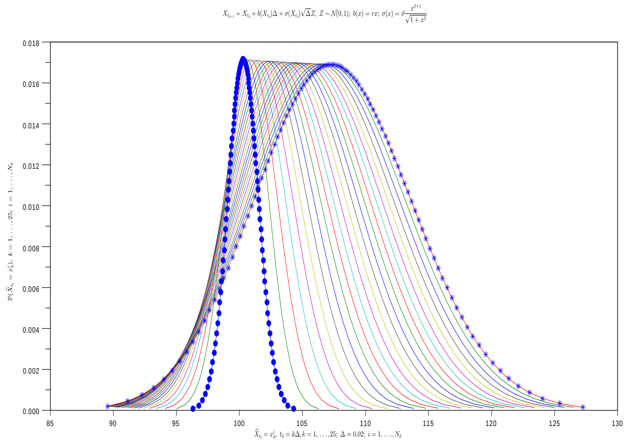


Figure: ("Pseudo-CEV model") $dX_t = rX_t dt + \vartheta(X_t^{\delta+1}/(1+X_t^2)^{-1/2})dW_t$, $X_0 = 100,\ r = 0.15,\ \vartheta = 0.7,\ T = 0.5$. Optimal grids, $\hat{X}_{t_k} = x_k^i,\ t_k = k\Delta$, $\Delta = 0.02,\ k = 1,\ldots,25,\ i = 1,\ldots,N_k$ vs the associated weights.

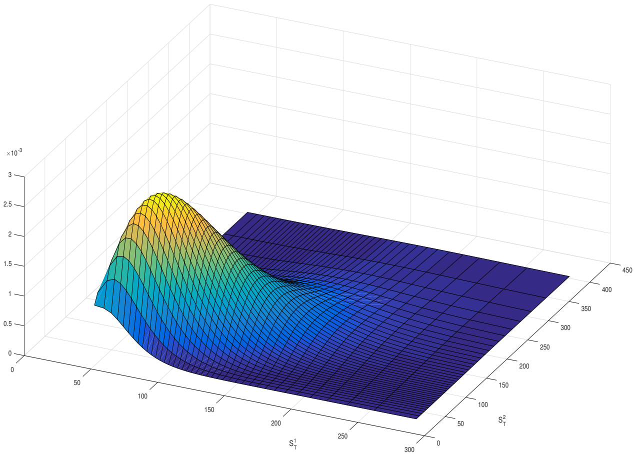


Figure: $\begin{cases} dS^1_t = rS^1_t + \rho\,\sigma_1 S^1_t dW^1_t + \sqrt{1-\rho^2}\,\sigma_1 S^1_t dW^2_t \\ dS^2_t = rS^2_t dt + \sigma_2 S^2_t dW^1_t \end{cases}$

$r = 0.04,\ \sigma_1 = 0.3,\ \sigma_2 = 0.4,\ \rho = 0.5,\ S^1_0 = 100,\ S^2_0 = 100,\ T = 1,\ n = 20$

# Plan

# Motivations

We want to compute $\mathbb{E}(f(X_T))$ (or $\mathbb{E}(f(X_{t_{k+1}})|X_{t_k})$) where $X$ is a solution to the SDE

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

where $W$ is a standard $q$-dimensional BM, ind. from $X_0$, $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times q}$ are Borel measurable functions and satisfy appropriate conditions. The quantities of interest have in general no explicit solution.

Then, $\mathbb{E}f(X_T)$ e.g. have to be approximated, for example, by

$$\mathbb{E}\big[f(\bar{X}_T)\big] \tag{1}$$

where $(\bar{X}_{t_k})_{k=0,\ldots,n}$ is a discretization scheme of the process $(X_t)_{t\geq 0}$ on $[0, T]$, for a given discretization mesh $t_k = k\Delta$, $k = 0, \ldots, n$, $\Delta = T/n$:

$$\begin{aligned} \bar{X}_{t_{k+1}} &= \bar{X}_{t_k} + b(t_k, \bar{X}_{t_k})\Delta + \sigma(t_k, \bar{X}_{t_k})(W_{t_{k+1}} - W_{t_k}), \ \bar{X}_0 = X_0 \\ &= \mathcal{E}_k(\bar{X}_{t_k}, Z_{k+1}), \qquad Z_{k+1} \sim \mathcal{N}(0, I_d). \end{aligned}$$

At this stage, the quantity (1) still has no closed formula in the general setting so that we have to make a spacial approximation of the expectation or the conditional expectation.

- This may be done by Monte Carlo simulation techniques or by optimal quantization method (Using for example stochastic algorithms or the recursive quantization (see Pagès and Sagna)).
- The aim of this work is to present another approach to quantize the Euler scheme of an $\mathbb{R}^d$-valued diffusion process in order to speak of fast only quantization in dimension greater than one.
- We propose a Markovian and product quantization method. It allows us to compute very quickly (in seconds order) the optimal product quantizers and its companion weights and transition probabilities when the size of the quantizations are chosen reasonably.

# Plan

Motivations

Short overview on the optimal quantization

Markovian product quantization

Application

## Optimal vector quantization

▷ Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $X : (\Omega, \mathcal{A}, \mathbb{P}) \longrightarrow \mathbb{R}^d$ be a r.v. with distribution $\mathbb{P}_X$. Assume that $X \in L^r(\mathbb{P})$

▷ The $L^r$-optimal quantization problem at level $N$ for $X$ consists in finding the best approximation of $X$ by a Borel function $\pi(X)$ of $X$ taking at most $N$ values.

▷ We associate to every Borel function $\pi(X)$ taking at most $N$ values, the $L^r$-mean error $(\mathbb{E}|X - \pi(X)|^r)^{1/r}$, where $|\cdot|$ denotes an arbitrary norm on $\mathbb{R}^d$.

▷ Then finding the best approximation of $X$ by a Borel function of $X$ taking at most $N$ values turns out to solve :

$$e_{N,r}(X) = \inf\{\|X - \pi(X)\|_r, \pi : \mathbb{R}^d \to \Gamma, \Gamma \subset \mathbb{R}^d, |\Gamma| \le N\},$$

## Optimal vector quantization

▷ Let $\Gamma = \{x_1, \cdots, x_N\} \subset \mathbb{R}^d$ be a an $N$-quantizer (or a grid of size $N$) and define a Voronoi partition $(C_i(\Gamma))_{i=1,\cdots,N}$ of $\mathbb{R}^d$: $\forall i$,

$$C_i(\Gamma) \subset \left\{x \in \mathbb{R}^d : |x - x_i| = \min_{j=1,\cdots,N} |x - x_j|\right\}.$$

▷ Consider the quantization of $X$ by the $N$-quantizer $\Gamma$, defined by

$$\widehat{X}^\Gamma = \sum_{i=1}^{N} x_i \mathbf{1}_{\{X \in C_i(\Gamma)\}} = \mathrm{Proj}_\Gamma(X). \tag{2}$$

▷ Then, $e_{N,r}(X)$ reads ($\|Y\|_r = (\mathbb{E}|Y|^r)^{1/r}$ for every $Y \in L^r(\mathbb{P})$)

$$e_{N,r}(X) = \inf\{\|X - \widehat{X}^\Gamma\|_r, \Gamma \subset \mathbb{R}^d, |\Gamma| \leq N\} \tag{3}$$

▷ For every $N \geq 1$, the infimum in (3) is attained at one $N$-quantizer (an $L^r$-optimal $N$-quantizer) at least. When $|\mathrm{supp}(\mathbb{P}_X))| \geq N$, any $L^r$-optimal $N$-quantizer has size $N$ (see Graf-Luschgy/Pagès). The quantization error, $e_{N,r}(X)$, decreases to zero as $N$ goes to infinity: Zador Theorem.

## Zador theorem

### Theorem

*(a) (Zador/Graf-Luschgy). Let $X$ be an $\mathbb{R}^d$-valued r.v. s.t. $\mathbb{E}|X|^{r+\eta} < +\infty$, $\eta > 0$ and let $\mathbb{P}_X = f \cdot \lambda_d + P_s$. Then*

$$\lim_{N \to +\infty} N^{\frac{1}{d}} e_{N,r}(X) = \widetilde{Q}_r(\mathbb{P}_X) \tag{4}$$

*with* $\widetilde{Q}_r(\mathbb{P}_X) = \left( \int_{\mathbb{R}^d} f^{\frac{d}{d+r}} d\lambda_d \right)^{\frac{1}{r}+\frac{1}{d}} \inf_{N \geq 1} N^{\frac{1}{d}} e_{N,r}(U([0,1]^d))$,

*(b) (Pierce/GraLus-LusPag). Let $\eta > 0$. There exists an universal constant $K_{2,d,\eta}$ s.t. for every r.v. $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d$,*

$$\inf_{|\Gamma| \leq N} \|X - \widehat{X}^{\Gamma}\|_2 \leq K_{2,d,\eta}\, \sigma_{2,\eta}(X) N^{-\frac{1}{d}}, \tag{5}$$

*where*

$$\sigma_{2,\eta}(X) = \inf_{\zeta \in \mathbb{R}^d} \|X - \zeta\|_{2+\eta} \leq +\infty.$$

## Distortion function

Define the distortion function for every $\Gamma = (x_1, \ldots, x_N)$ by

$$D_{N,2}(\Gamma) = \mathbb{E}|X - \widehat{X}^{\Gamma}|^2 = \sum_{i=1}^{N} \int_{C_i(\Gamma)} |x - x_i|^2 d\mathbb{P}_X(x), \qquad (6)$$

so that $e^2_{N,2}(X) = \inf_{\Gamma \in (\mathbb{R}^d)^N} D_{N,2}(\Gamma)$.

### Proposition

*$D_{N,2}$ is differentiable at any $N$-tuple $\Gamma \in (\mathbb{R}^d)^N$ having pairwise distinct components and such that $\mathbb{P}\left(X \in \cup_i \partial C_i(\Gamma)\right) = 0$, and,*

$$\nabla D_{N,2}(\Gamma) = 2\Big(\int_{C_i(\Gamma)} (x_i - x) d\mathbb{P}_X(x)\Big)_{i=1,\cdots,N}. \qquad (7)$$

For numerics, the search of optimal (or stationary) quantizers is based on zero search recursive procedures like Newton-Raphson algorithm for real valued r.v. and other algorithms when $d \geq 2$. Optimal $\mathcal{N}(0; I_d)$ grids available at www.quantize.math-fi.com.

## Error Analysis

Error approximation of $\mathbb{E}f(X)$ by $\mathbb{E}f(\widehat{X}^{\Gamma})$: (see Pagès-Printems).

(a) Let $\Gamma$ be a stationary quantizer and $f$ be a Borel function on $\mathbb{R}^d$. If $f$ is a convex function then

$$\mathbb{E}f(\widehat{X}^{\Gamma}) \leq \mathbb{E}f(X).$$

(b) Lipschitz functions:

- If $f$ is Lipschitz continuous then for any $N$-quantizer $\Gamma$ we have

$$\left|\mathbb{E}f(X) - \mathbb{E}f(\widehat{X}^{\Gamma})\right| \leq [f]_{\mathrm{Lip}} \|X - \widehat{X}^{\Gamma}\|_{2},$$

- Let $\theta : \mathbb{R}^d \to \mathbb{R}_+$ be a nonnegative convex function such that $\theta(X) \in L^2(\mathbb{P})$. If $f$ is locally Lipschitz with at most $\theta$-growth, i.e. $|f(x) - f(y)| \leq [f]_{\mathrm{Lip}}|x - y|(\theta(x) + \theta(y))$ then $f(X) \in L^1(\mathbb{P})$ and

$$\left|\mathbb{E}f(X) - \mathbb{E}f(\widehat{X}^{\Gamma})\right| \leq 2[f]_{\mathrm{Lip}} \|X - \widehat{X}^{\Gamma}\|_{2} \|\theta(X)\|_{2}.$$

(c) Differentiable functionals: if $f$ is differentiable on $\mathbb{R}^d$ with an $\alpha$-Hölder gradient $\nabla f$ ($\alpha \in [0, 1]$), then for any stationary $\Gamma$,

$$\left|\mathbb{E}f(X) - \mathbb{E}f(\widehat{X}^{\Gamma})\right| \leq [\nabla f]_{\alpha} \|X - \widehat{X}^{\Gamma}\|_{2}^{1+\alpha}.$$

## The recursive quantization of the Euler scheme (Pagès and Sagna)

In practice, the recursive quantization of the Euler scheme $(\bar{X}_{t_k})$ consists to compute a sequence $(\Gamma_k)$ of quantizers defined by

$$\Gamma_k \in \arg\min\{\bar{D}_k(\Gamma), \Gamma \subset \mathbb{R}^d, \mathrm{card}(\Gamma) \leq N_k\}$$

where $\bar{D}_k(\cdot)$ is the distortion associated to $\bar{X}_{t_k}$ and defined by

$$\bar{D}_k(\Gamma_k) = \mathbb{E}\,\mathrm{dist}\big(\bar{X}_{t_k}, \Gamma_k\big)^2 = \mathbb{E}\big[\mathrm{dist}(\mathcal{E}_{k-1}(\bar{X}_{t_{k-1}}, Z_k), \Gamma_k)^2\big]. \quad (8)$$

▷ *Recursive (marginal) quantization method.* We quantize $\bar{X}_0$ by $\widehat{X}_0^{\Gamma_0}$. To define the recursive quantization of $\bar{X}_{t_1}$ we replace $\bar{X}_0$ by $\widehat{X}_0^{\Gamma_0}$ in (8), then, we set $\widetilde{X}_{t_1} := \mathcal{E}_0(\widehat{X}_0^{\Gamma_0}, Z_1)$ and consider the induced distortion

$$\widetilde{D}_1(\Gamma) := \mathbb{E}\big[\mathrm{dist}(\widetilde{X}_{t_1}, \Gamma)^2\big] = \mathbb{E}\big[\mathrm{dist}(\mathcal{E}_0(\widehat{X}_0^{\Gamma_0}, Z_1), \Gamma)^2\big],$$

where $\Gamma \subset \mathbb{R}^d$ and $\mathrm{card}(\Gamma) \leq N_1$.

## The recursive quantization of the Euler scheme (Pagès and Sagna)

⇝ The distortion function $\widetilde{D}_1(\cdot)$ is the one to be optimized in order to produce the optimal $N_1$-quantizer $\Gamma_1$.

⇝ Consequently, we define the recursive marginal quantization of $\bar{X}_{t_1}$ as the optimal quantization of $\widetilde{X}_{t_1}$: $\widehat{X}_{t_1}^{\Gamma_1} = \mathrm{Proj}_{\Gamma_1}(\widetilde{X}_{t_1})$, where

$$\Gamma_1 \in \arg\min\{\widetilde{D}_1(\Gamma),\ \Gamma \subset \mathbb{R}^d,\ \mathrm{card}(\Gamma) \leq N_1\}.$$

⇝ Once the optimal $N_1$-quantizer $\Gamma_1$ is produced, we define the recursive quantization of $\bar{X}_{t_2}$ as the OQ $\widehat{X}_{t_2}^{\Gamma_2}$ of $\widetilde{X}_{t_1}$ where

$$\Gamma_2 \in \arg\min\{\widetilde{D}_2(\Gamma),\ \Gamma \subset \mathbb{R}^d,\ \mathrm{card}(\Gamma) \leq N_2\}$$
$$\widetilde{D}_2(\Gamma) = \mathbb{E}\big[\mathrm{dist}(\widetilde{X}_{t_2}, \Gamma)^2\big] \quad \text{and} \quad \widetilde{X}_{t_2} := \mathcal{E}_1(\widehat{X}_1^{\Gamma_1}, Z_2).$$

⇝ Repeating this procedure, we define the recursive quantization of $(\bar{X}_{t_k})_{0\leq k\leq n}$ as the optimal quantizations $(\widehat{X}_{t_k}^{\Gamma_k})_{0\leq k\leq n}$ of the process $(\widetilde{X}_{t_k})_{0\leq k\leq n}$: $\forall\, k \in \{0,\dots,n\}$, $\widehat{X}_{t_k}^{\Gamma_k} = \mathrm{Proj}_{\Gamma_k}(\widetilde{X}_{t_k})$, with $\widetilde{X}_0 = \bar{X}_0$.

## The recursive quantization of the Euler scheme (Pagès and Sagna)

$\rightsquigarrow$ This leads us to consider the sequence of recursive marginal quantizations $(\widehat{X}^{\Gamma_k}_{t_k})_{k=0,\ldots,N}$ of $(\bar{X}_{t_k})_{k=0,\ldots,N}$, defined from the following recursion:

$$\begin{aligned} \widetilde{X}_0 &= \bar{X}_0 \\ \widehat{X}^{\Gamma_k}_{t_k} &= \mathrm{Proj}_{\Gamma_k}(\widetilde{X}_{t_k}) \text{ and } \widetilde{X}_{t_{k+1}} = \mathcal{E}_k(\widehat{X}^{\Gamma_k}_{t_k}, Z_{k+1}),\ k = 0, \ldots, n-1 \end{aligned}$$

where $(Z_k)_{k=1,\ldots,n}$ is an i.i.d. sequence of $\mathcal{N}(0; I_q)$-distributed random vectors, independent of $\bar{X}_0$.

$\triangleright$ From an analytical point of view, we show in particular that for any sequence $(\widehat{X}^{\Gamma_k}_{t_k})_{0\le k\le n}$ of (quadratic) optimal recursive quantization of $(\bar{X}_{t_k})_{0\le k\le n}$, the quantization error $\|\bar{X}_{t_k} - \widehat{X}^{\Gamma_k}_{t_k}\|_2$, at the step $k$ of the recursion is given for any $\eta \in ]0,1]$ by

$$\|\bar{X}_k - \widehat{X}^{\Gamma_k}_k\|_2 \le \sum_{\ell=0}^{k} a_\ell N_\ell^{-1/d},$$

where $a_\ell$ is a positive real constant depending on $b$, $\sigma$, $\Delta$, $x_0$, $\eta$

# Plan

## Markovian product quantization: description of the method

$\rightsquigarrow$ Denote by $\Gamma_k^\ell$ an $N_k^\ell$-quantizer of the $\ell$-th component $\bar{X}_k^\ell$ of the vector $\bar{X}_k$ and let $\widehat{X}_k^\ell$ be the quantization of $\bar{X}_k^\ell$ of size $N_k^\ell$, on the grid $\Gamma_k^\ell$.

$\rightsquigarrow$ Define the product quantizer $\Gamma_k = \bigotimes_{\ell=1}^d \Gamma_k^\ell$ (of the vector $\bar{X}_k$) of size $N_k = N_k^1 \times \ldots \times N_k^d$ as

$$\Gamma_k = \{(x_k^{1,i_1}, \ldots, x_k^{d,i_d}), \quad i_\ell \in \{1, \ldots, N_k^\ell\},\ \ell \in \{1, \ldots, d\}\}.$$

$\rightsquigarrow$ Set, for every $k \in \{0, \ldots, n\}$,

$$\mathscr{I}_k = \left\{(i_1, \ldots, i_d),\ i_\ell \in \{1, \ldots, N_k^\ell\}\right\} \tag{9}$$

and for $i := (i_1, \ldots, i_d) \in \mathscr{I}_k$, set

$$x_k^i := (x_k^{1,i_1}, \ldots, x_k^{d,i_d}). \tag{10}$$

$\rightsquigarrow$ To define the Markovian product quantization, suppose that $\bar{X}_k$ has already been quantized and that we have access to the companions probabilities $\mathbb{P}(\widehat{X}_k = x_k^i)$, $i \in \mathscr{I}_k$.

## Markovian product quantization: description of the method

⇝ Setting $\widetilde{X}^\ell_{k+1} = \mathcal{E}^\ell_k(\widehat{X}_k, Z_{k+1})$. We may approximate the distortion function $\bar{D}^\ell_{k+1}$ associated to the $\ell$-th component of the vector $\bar{X}^\ell_{k+1}$ by

$$\begin{aligned}\widetilde{D}^\ell_{k+1}(\Gamma^\ell_{k+1}) &:= \mathbb{E}\big[\mathrm{dist}(\widetilde{X}^\ell_{k+1}, \Gamma^\ell_{k+1})^2\big] \\ &= \mathbb{E}\big[\mathrm{dist}(\mathcal{E}^\ell_k(\widehat{X}_k, Z_{k+1}), \Gamma^\ell_{k+1})^2\big] \\ &= \sum_{i \in \mathscr{I}_k} \mathbb{E}\big[\mathrm{dist}(\mathcal{E}^\ell_k(x^i_k, Z_{k+1}), \Gamma_{k+1})^2\big]\mathbb{P}\big(\widehat{X}_k = x^i_k\big).\end{aligned}$$

This allows us to consider the sequence of product recursive quantizations $(\widehat{X}_k)_{k=0,\cdots,n}$ of $(\bar{X}_k)_{k=0,\cdots,n}$, defined from the following recursion for every $k = 0, \ldots, n-1$:

$$\begin{cases} \widetilde{X}_0 = \widehat{X}_0, \quad \widehat{X}^\ell_k = \mathrm{Proj}_{\Gamma^\ell_k}(\widetilde{X}^\ell_k),\ \ell = 1, \ldots, d \\ \widehat{X}_k = (\widehat{X}^1_k, \ldots, \widehat{X}^d_k) \quad \text{and} \quad \widetilde{X}^\ell_{k+1} = \mathcal{E}^\ell_k(\widehat{X}_k, Z_{k+1}),\ \ell = 1, \ldots, d \\ \mathcal{E}^\ell_k(x, z) = m^\ell_k(x) + \sqrt{\Delta}(\sigma^{\ell\bullet}(t_k, x)|z),\ m^\ell_k(x) = x^\ell + \Delta b^\ell(t_k, x) \\ z = (z^1, \ldots, z^q) \in \mathbb{R}^q,\ x = (x^1, \ldots, x^d),\ b = (b^1, \ldots, b^d) \end{cases}$$

## Markov property

**Remark.** The process $(\widehat{X}_k)_{k\geq 0}$ is a Markov chain on $\mathbb{R}^d$.

In fact, setting $\mathcal{F}_k^{\widehat{X}} = \sigma(\widehat{X}_0, \ldots, \widehat{X}_k)$, we have for any bounded function $f : \mathbb{R}^d \to \mathbb{R}$

$$\begin{aligned}\mathbb{E}(f(\widehat{X}_{k+1})|\mathcal{F}_k^{\widehat{X}}) &= \sum_{j\in\mathscr{I}_{k+1}} \mathbb{E}\left(f(x_{k+1}^j)\mathbb{1}_{\{\widehat{X}_{k+1}=x_{k+1}^j\}}|\mathcal{F}_k^{\widehat{X}}\right)\\ &= \sum_{j\in\mathscr{I}_{k+1}} f(x_{k+1}^j)\mathbb{E}\left(\mathbb{1}_{\{\mathcal{E}_k(\widehat{X}_k,Z_{k+1})\in\prod_{\ell=1}^d C_{j_\ell}(\Gamma_{k+1}^\ell)\}}|\mathcal{F}_k^{\widehat{X}}\right)\end{aligned}$$

where $\mathcal{E}_k(\widehat{X}_k, Z_{k+1}) = (\mathcal{E}_k^1(\widehat{X}_k, Z_{k+1}), \ldots, \mathcal{E}_k^d(\widehat{X}_k, Z_{k+1}))$. So that

$$\mathbb{E}(f(\widehat{X}_{k+1})|\mathcal{F}_k^{\widehat{X}}) = \sum_{j\in\mathscr{I}_{k+1}} f(x_{k+1}^j)h_j(\widehat{X}_k),$$

where for every $x \in \mathbb{R}^d$,

$$h_j(x) = \mathbb{P}\big(\mathcal{E}_k(x, Z_{k+1}) \in \prod_{\ell=1}^d C_{j_\ell}(\Gamma_{k+1}^\ell)\big).$$

## The companion weights and transition probabilities

Let us set, for every $k \in \{0, \dots, n-1\}$ and for every $j \in \mathscr{I}_{k+1}$,

$$x_{k+1}^{\ell, j_\ell - 1/2} = \frac{x_{k+1}^{\ell, j_\ell} + x_{k+1}^{\ell, j_\ell - 1}}{2}, \qquad x_{k+1}^{\ell, j_\ell + 1/2} = \frac{x_{k+1}^{\ell, j_\ell} + x_{k+1}^{\ell, j_\ell + 1}}{2}$$

$\forall\, x \in \mathbb{R}^d$: $\vartheta_k^\ell(x)^2 = \sum_{p=1}^q \Delta\big(\sigma_k^{\ell p}(x)\big)^2 = \Delta\, |\sigma_k^{\ell \bullet}(x)|_2^2$ and

if $Z_k^{(2:q)} = z \in \mathbb{R}^{q-1}$ and $x \in \mathbb{R}^d$, we set (if $\sigma_k^{\ell 1}(x) > 0$)

$$x_{k+1}^{\ell, j_\ell -}(x, z) := \frac{x_{k+1}^{\ell, j_\ell - 1/2} - m_k^\ell(x) - \sqrt{\Delta}\big(\sigma_k^{(\ell, 2:q)}(x) | z\big)}{\sqrt{\Delta}\sigma_k^{\ell 1}(x)}$$

$$\text{and} \quad x_{k+1}^{\ell, j_\ell +}(x, z) := \frac{x_{k+1}^{\ell, j_\ell + 1/2} - m_k^\ell(x) - \sqrt{\Delta}\big(\sigma_k^{(\ell, 2:q)}(x) | z\big)}{\sqrt{\Delta}\sigma_k^{\ell 1}(x)}.$$

We also set

$$\mathbb{J}_{k, j_\ell}^0(x) = \Big\{ z \in \mathbb{R}^{q-1},\ \sqrt{\Delta}\big(\sigma_k^{(\ell, 2:q)}(x) | z\big) \in \big(x_{k+1}^{\ell, j_\ell - 1/2} - m_k^\ell(x), x_{k+1}^{\ell, j_\ell + 1/2} - m_k^\ell(x)\big)$$

and

## The companion weights and transition probabilities

$$\mathbb{J}_k^0(x) = \{\ell \in \{1,\dots,d\}, \quad \sigma_k^{\ell 1}(x) = 0\}$$
$$\mathbb{J}_k^-(x) = \{\ell \in \{1,\dots,d\}, \quad \sigma_k^{\ell 1}(x) < 0\}$$
$$\mathbb{J}_k^+(x) = \{\ell \in \{1,\dots,d\}, \quad \sigma_k^{\ell 1}(x) > 0\}.$$

Proposition. Let $\{\widehat{X}_k, k = 0,\dots,n\}$ be the sequence of Markovian product quantization. Then, $\mathbb{P}(\widehat{X}_{k+1} = x_{k+1}^j | \widehat{X}_k = x_k^i)$ equals

$$\mathbb{E} \prod_{\ell \in \mathbb{J}_k^0(x_k^i)} \mathbf{1}_{\{\zeta \in \mathbb{J}^0_{k,j_\ell}(x_k^\ell)\}} \max\left(\Phi_0(\beta_j(x_k^i,\zeta)) - \Phi_0(\alpha_j(x_k^i,\zeta)), 0\right)$$

where $\zeta \sim \mathcal{N}(0; I_{q-1})$ and where for every $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^{q-1}$,

$$\alpha_j(x,z) = \max\Big(\sup_{\ell \in \mathbb{J}_k^+(x)} x_{k+1}^{\ell, j_\ell -}(x,z), \sup_{\ell \in \mathbb{J}_k^-(x)} x_{k+1}^{\ell, j_\ell +}(x,z)\Big)$$

$$\text{and} \quad \beta_j(x,z) = \min\Big(\inf_{\ell \in \mathbb{J}_k^+(x)} x_{k+1}^{\ell, j_\ell +}(x,z), \inf_{\ell \in \mathbb{J}_k^-(x)} x_{k+1}^{\ell, j_\ell -}(x,z)\Big),$$

## Weights and transition probabilities of $\widehat{X}_k^\ell$

⇝ In the particular case where the volatility matrix $\sigma(t,x)$ of $(X_t)_{t\geq 0}$ is a diagonal matrix with positive diagonal terms

$$\hat{p}_k^{ij} = \prod_{\ell=1}^{d} \left[\Phi_0\big(x_{k+1}^{\ell,j_\ell+}(x_k^i,0)\big) - \Phi_0\big(x_{k+1}^{\ell,j_i-}(x_k^i,0)\big)\right].$$

Proposition. 1. For any $\ell \in \{1,\dots,d\}$ and any $j_\ell \in \{1,\dots,N_{k+1}^\ell\}$,

$$\mathbb{P}\big(\widetilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}^\ell)|\widehat{X}_k = x_k^i\big) = \Phi_0\big(x_{k+1}^{\ell,j_\ell+}(x_k^i,0)\big) - \Phi_0\big(x_{k+1}^{\ell,j_\ell-}(x_k^i,0)\big).$$

Remark. We remark that

⇝ This allows us to compute the weights $\mathbb{P}\big(\widetilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}^\ell)\big)$.

⇝ For $\ell,\ell' \in \{1,\dots,d\}$, $j_\ell \in \{1,\dots,N_{k+1}^\ell\}$, $j_{\ell'} \in \{1,\dots,N_k^{\ell'}\}$,

$$\mathbb{P}\big(\widehat{X}_{k+1}^\ell = x_{k+1}^{\ell,j_\ell}|\widehat{X}_k^{\ell'} = x_k^{\ell',j_{\ell'}}\big) = \sum_{i\in\mathscr{I}_k} \delta_{\{j_{\ell'}=i_{\ell'}\}} \frac{\hat{p}_k^{ij_\ell}}{\hat{p}_k^{j_{\ell'}}} \mathbb{P}(\widehat{X}_k = x_k^i).$$

## Computing the Markovian product quantizers

Recall that for every $\ell = 1, \ldots, d$, for every $k = 0, \ldots, n-1$,

$$\widetilde{D}^{\ell}_{k+1}(\Gamma^{\ell}_{k+1}) = \sum_{i \in \mathscr{I}_k} \mathbb{E}\big[d(\mathcal{E}^{\ell}_k(x^i_k, Z_{k+1}), \Gamma^{\ell}_{k+1})^2\big]\mathbb{P}\big(\widehat{X}_k = x^i_k\big).$$

$\widetilde{D}^{\ell}_{k+1}(\Gamma^{\ell}_{k+1})$ is continuously differentiable as a function of the $N_{k+1}$-quantizer $\Gamma^{i}_{k+1}$ (having pairwise distinct components) and its gradient vector components read

$$\frac{\partial \widetilde{D}^{\ell}_{k+1}(\Gamma^{\ell}_{k+1})}{\partial x^{\ell, j_\ell}_{k+1}} = \sum_{i \in \mathscr{I}_k} \Psi'_{j_\ell}(x^i_k)\, p^i_k = \mathbb{E}\Psi'_{j_\ell}(\widehat{X}_k), \qquad (11)$$

where for every $x \in \mathbb{R}^d$,

$$\begin{aligned}\Psi'_{j_\ell}(x) \;&=\; \big(x^{\ell, j_\ell}_{k+1} - m^{\ell}_k(x)\big)\Big(\Phi_0\big(x^{\ell, j_\ell +}_{k+1}(x)\big) - \Phi_0\big(x^{\ell, j_\ell -}_{k+1}(x)\big)\Big) \\ &\quad + \vartheta^{\ell}_k(x)\Big(\Phi'_0\big(x^{\ell, j_\ell +}_{k+1}(x)\big) - \Phi'_0\big(x^{\ell, j_\ell -}_{k+1}(x)\big)\Big).\end{aligned}$$

The sub-diagonal, the super-diagonals and the diagonal terms of the Hessian matrix are given respectively by

$$\frac{\partial^2 \widetilde{D}^{\ell}_{k+1}(\Gamma^{\ell}_{k+1})}{\partial x^{\ell,j_\ell}_{k+1}\partial x^{\ell,j_\ell-1}_{k+1}} = \sum_{i\in\mathscr{I}_k} \Psi''_{j_\ell,j_\ell-1}(x^i_k)\, p^i_k = \mathbb{E}\Psi''_{j_\ell,j_\ell-1}(\widehat{X}_k),$$

$$\frac{\partial^2 \widetilde{D}^{\ell}_{k+1}(\Gamma^{\ell}_{k+1})}{\partial x^{\ell,j_\ell}_{k+1}\partial x^{\ell,j_\ell+1}_{k+1}} = \sum_{i\in\mathscr{I}_k} \Psi''_{j_\ell,j_\ell+1}(x^i_k)\, p^i_k = \mathbb{E}\Psi''_{j_\ell,j_\ell+1}(\widehat{X}_k),$$

$$\text{and}\qquad \frac{\partial^2 \widetilde{D}^{\ell}_{k+1}(\Gamma^{\ell}_{k+1})}{\partial^2 x^{\ell,j_\ell}_{k+1}} = \sum_{i\in\mathscr{I}_k} \Psi''_{j_\ell,j_\ell}(x^i_k)\, p^i_k = \mathbb{E}\Psi''_{j_\ell,j_\ell}(\widehat{X}_k),$$

where for every $x \in \mathbb{R}^d$,

$$\Psi''_{j_\ell,j_\ell-1}(x) = -\frac{1}{4}\frac{1}{\vartheta^{\ell}_k(x)}(x^{\ell,j_\ell}_{k+1} - x^{\ell,j_\ell-1}_{k+1})\Phi'_0\big(x^{\ell,j_\ell-}_{k+1}(x)\big),$$

$$\Psi''_{j_\ell,j_\ell+1}(x) = -\frac{1}{4}\frac{1}{\vartheta^{\ell}_k(x)}(x^{\ell,j_\ell+1}_{k+1} - x^{\ell,j_\ell}_{k+1})\Phi'_0\big(x^{\ell,j_\ell+}_{k+1}(x)\big),$$

$$\Psi''_{j_\ell,j_\ell}(x) = \Phi_0\big(x^{\ell,j_\ell+}_{k+1}(x)\big) - \Phi_0\big(x^{\ell,j_\ell-}_{k+1}(x)\big) + \Psi''_{j_\ell,j_\ell-1}(x) + \Psi''_{j_\ell,j_\ell+}$$

## Newton and Lloyd algorithms

⇝ Once we have access to $\nabla \widetilde{D}^{\ell}_{k+1}$ and $\nabla^2 \widetilde{D}^{\ell}_{k+1}$ we may write down a Newton-Raphson zero search procedure to compute $\Gamma^{\ell}_{k+1}$. It is indexed by $p \geq 0$, where a current grid $\Gamma^{\ell,p}_{k+1}$ is updated as:

$$\Gamma^{\ell,p+1}_{k+1} = \Gamma^{\ell,p}_{k+1} - \big(\nabla^2 \widetilde{D}^{\ell}_{k+1}(\Gamma^{\ell,p}_{k+1})\big)^{-1} \nabla \widetilde{D}^{\ell}_{k+1}(\Gamma^{\ell,p}_{k+1}), \quad p \geq 1,$$

starting from a $\Gamma^{\ell,0}_{k+1} \in \mathbb{R}^{N^{\ell}_{k+1}}$ (with increasing components).

⇝ If $\Gamma^{\ell}_{k+1} = \{x^{\ell,j_\ell}_{k+1}, j_\ell = 1, \ldots, N^{\ell}_{k+1}\}$ is an optimal Markovian product quantizer for $\widetilde{X}^{\ell}_{k+1}$ then it is a stationary quantizer for $\widetilde{X}^{\ell}_{k+1}$, means, $\mathbb{E}\Big(\widetilde{X}^{\ell}_{k+1} \big| \widehat{X}^{\ell}_{k+1}\Big) = \widehat{X}^{\ell}_{k+1}$. Then

$$x^{\ell,j_\ell}_{k+1} = \frac{\sum_{i \in \mathscr{I}_k} \left[m^{\ell}_k(x^i_k)\gamma_{\ell,k}(x^i_k) - \vartheta^{\ell}_k(x^i_k)\gamma'_{\ell,k}(x^i_k)\right] p^i_k}{\sum_{i \in \mathscr{I}_k} \gamma_{\ell,k}(x^i_k)\, p^i_k} \qquad (12)$$

where for every $x \in \mathbb{R}^d$,

$$\gamma_{\ell,k}(x) = \Phi_0\Big(x^{\ell,j_\ell+}_{k+1}(x)\Big) - \Phi_0\Big(x^{\ell,j_\ell-}_{k+1}(x)\Big), \; \gamma'_{\ell,k}(x) = \Phi'_0\Big(x^{\ell,j_\ell+}_{k+1}(x)\Big) - \Phi'_0\Big(x^{\ell,j_\ell-}_{k+1}(x)\Big).$$

# Error Analysis

Suppose that

$$|b(t,x) - b(t,y)| \leq [b]_{\mathrm{Lip}}|x-y| \tag{13}$$

$$\|\sigma(t,x) - \sigma(t,y)\| \leq [\sigma]_{\mathrm{Lip}}|x-y| \tag{14}$$

$$|b(t,x)| \leq L(1+|x|) \ \text{ and } \ \|\sigma(t,x)\| \leq L(1+|x|). \tag{15}$$

Theorem. Let the coefficients $b$, $\sigma$ satisfy the assumptions (13), (14) and (15). Let for every $k = 0, \cdots, n$, $\Gamma_k$ be a quadratic MP quantizer for $\widetilde{X}_k$ at level $N_k$. Then, $\forall k = 0, \cdots, n$, $\forall\, \eta \in ]0,1]$,

$$\|\bar{X}_k - \widehat{X}_k^{\Gamma_k}\|_2 \leq K_{2,\eta} \sum_{\ell=1}^{k} e^{(k-\ell)\Delta C_{b,\sigma}} a_\ell(\cdot,\ldots,\cdot) \Big( \sum_{i=1}^{d} (N_\ell^i)^{-2/d} \Big)^{1/2}$$

where for every $p \in (2,3]$,

$$a_\ell(\cdot) := e^{C_{b,\sigma}\frac{(t_k - t_\ell)}{p}} \left[ e^{(\kappa_p + K_p)t_\ell}|x_0|^p + d^{(k-1)(\frac{p}{2}-1)} \frac{e^{\kappa_p \Delta} L + K_p}{\kappa_p + K_p} \left( e^{(\kappa_p+K_p)t_\ell} - 1 \right) \right]^{\frac{1}{p}},$$

with $C_{b,\sigma} = [b]_{\mathrm{Lip}} + \frac{1}{2}[\sigma]^2_{\mathrm{Lip}}$, $K_{2,\eta}$ is a universal constant defined in the Pierce's Lemma;

$$\kappa_p := \left(\frac{(p+1)(p-2)}{2} + 2pL\right) \quad \text{and} \quad K_p := 2^{p-1} L^p \left(1 + p + \frac{p(p-1)}{2} \Delta^{\frac{p}{2}-1}\right) \mathbb{E}|Z|^p.$$

Notice that if we take the same grid size $N_\ell^i = N_\ell$, for every $i \in \{1, \ldots, d\}$, the error bound (26) becomes

$$\|\bar{X}_k - \widehat{X}_k^{\Gamma_k}\|_2 \leq K_{2,\eta} \sqrt{d} \sum_{\ell=0}^{k} a_\ell(b, \sigma, t_k, \Delta, x_0, L, 2+\eta) N_\ell^{-1/d}.$$

# Plan

## BSDE

⇝ Consider the following Markovian BSDE

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T], \quad (16)$$

where $W$ is a $q$-dimensional BM, $Z \in \mathbb{R}^q$ is a square integrable progressively measurable process, $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}$. We suppose $\xi = h(X_T)$, where $X$ is a strong solution to the SDE

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \qquad x \in \mathbb{R}^d. \quad (17)$$

⇝ The discrete time quantized BSDE process $(\widehat{Y}_k)_{k=0,\cdots,n}$:

$$\begin{aligned} \widehat{Y}_n &= h(\widehat{X}_n) \\ \widehat{Y}_k &= \widehat{\mathbb{E}}_k(\widehat{Y}_{k+1}) + \Delta_n f_k\big(\widehat{X}_k, \widehat{\mathbb{E}}_k(\widehat{Y}_{k+1}), \widehat{\zeta}_k\big) \end{aligned}$$

with

$$\widehat{\zeta}_k = \frac{1}{\Delta_n}\widehat{\mathbb{E}}_k(\widehat{Y}_{k+1}\Delta W_{t_{k+1}}), k = 0, \cdots, n-1,$$

where $\widehat{\mathbb{E}}_k = \mathbb{E}(\cdot \,|\, \widehat{X}_k)$.

$\rightsquigarrow$ *Explicit numerical scheme for the BSDE.* For $i \in \mathscr{I}_k$, $j \in \mathscr{I}_{k+1}$,

$$p_k^i = \mathbb{P}(\widehat{X}_k = x_k^i), \;\; k = 0, \cdots, n$$
$$\text{and} \qquad p_k^{ij} = \mathbb{P}(\widehat{X}_{k+1} = x_{k+1}^j | \widehat{X}_k = x_k^i), \;\; k = 0, \cdots, n-1.$$

Setting $\widehat{Y}_k = \widehat{y}_k(\widehat{X}_k)$, for every $k \in \{0, \cdots, n\}$, the quantized BSDE scheme reads as

$$\begin{cases} \widehat{y}_n(x_n^i) = h(x_n^i) & x_n^i \in \Gamma_n \\ \widehat{y}_k(x_k^i) = \widehat{\alpha}_k(x_k^i) + \Delta_n f\big(t_k, x_k^i, \widehat{\alpha}_k(x_k^i), \widehat{\beta}_k(x_k^i)\big) & x_k \in \Gamma_k \end{cases}$$

where for $k = 0, \ldots, n-1$,

$$\widehat{\alpha}_k(x_k^i) = \sum_{j \in \mathscr{I}_{k+1}} \widehat{y}_{k+1}(x_{k+1}^j)\, p_k^{ij}, \;\; \widehat{\beta}_k(x_k^i) = \frac{1}{\sqrt{\Delta_n}} \sum_{j \in \mathscr{I}_{k+1}} \widehat{y}_{k+1}(x_{k+1}^j)\, \Lambda_k^{ij},$$

with

$$\Lambda_k^{ij} = \mathbb{E}\big(Z_{k+1} \mathbb{1}_{\{\widehat{X}_{k+1} = x_{k+1}^j\}} | \widehat{X}_k = x_k^i\big).$$

Proposition. Suppose $q = d$, $\mathcal{E}_k^\ell(x, Z_{k+1}) = \mathcal{E}_k^\ell(x, Z_{k+1}^\ell)$. Then

$$\Lambda_k^{ij,\ell} = \Big(\Phi_0'(x_{k+1}^{\ell,j_\ell-}(x_k^i)) - \Phi_0'(x_{k+1}^{\ell,j_\ell+}(x_k^i))\Big) \prod_{\ell'\neq\ell}^d \Big[\Phi_0\Big(x_{k+1}^{\ell',j_{\ell'}+}(x_k^i)\Big) - \Phi_0\Big(x_{k+1}^{\ell',j_{\ell'}-}(x_k^i)\Big)\Big].$$

In the general setting set

$$\mathbb{J}_{k,j_\ell}^{0,p}(x) = \Big\{z\in\mathbb{R}, \quad \sqrt{\Delta}\sigma_k^{\ell p}(x) z \in \Big(x_{k+1}^{\ell,j_\ell-1/2} - m_k^\ell(x), x_{k+1}^{\ell,j_\ell+1/2} - m_k^\ell(x)\Big)\Big\}$$

and

$$\mathbb{L}_k^{0,p}(x) = \Big\{\ell\in\{1,\dots,d\}, \quad \sum_{p'\neq p}\Big(\sigma_k^{\ell p'}(x)\Big)^2 = 0\Big\}.$$

We also set

$$x_{k+1}^{\ell,p,j_\ell-}(x,z) = \frac{x_{k+1}^{\ell,j_\ell-1/2} - m_k^\ell(x) - \sqrt{\Delta}\sigma_k^{\ell p}(x)z}{\sqrt{\Delta}\Big(\sum_{p'\neq p}\Big(\sigma_k^{\ell p'}(x)\Big)^2\Big)^{1/2}}; \quad x_{k+1}^{\ell,p,j_\ell+}(x,z) = \frac{x_{k+1}^{\ell,j_\ell+1/2} - m_k^\ell(x) - \sqrt{\Delta}\sigma_k^{\ell p}(x)z}{\sqrt{\Delta}\Big(\sum_{p'\neq p}\Big(\sigma_k^{\ell p'}(x)\Big)^2\Big)^{1/2}}.$$

Proposition. The $p$-th component $\Lambda_k^{ij,p}$ of $\Lambda_k^{ij}$ reads

$$\Lambda_k^{ij,p} = \mathbb{E}\,\zeta \prod_{\ell\in\mathbb{L}_k^{0,p}(x_k^i)} \mathbb{1}_{\{\zeta\in\mathbb{J}_{k,j_\ell}^{0,p}(x_k^i)\}} \Big(\Phi_0(\alpha_j^p(x_k^i,\zeta)) - \Phi_0(\beta_j^p(x_k^i,\zeta))\Big)^+$$

with the convention that $\prod_{\ell\in\emptyset}(\cdot)=1$, $\zeta\sim\mathcal{N}(0;1)$ and where for every $x\in\mathbb{R}^d$ and $z\in\mathbb{R}$,

$$\alpha_j^p(x,z)=\sup_{\ell\in\left(\mathbb{L}_k^{0,p}(x)\right)^c} x_{k+1}^{\ell,p,j_\ell-}(x,z),\ \beta_j^p(x,z)=\inf_{\ell\in\left(\mathbb{L}_k^{0,p}(x)\right)^c} x_{k+1}^{\ell,p,j_\ell+}(x,z).$$

In particular, if $p\in\{1,\ldots,q\}$ and if for every $\ell\in\{1,\ldots,d\}$ there exists $p'\neq p$ such that $\sigma_k^{\ell p'}(x)\neq 0$, then,

$$\Lambda_k^{ij,p}=\mathbb{E}\,\zeta\big(\Phi_0(\alpha_j^p(x_k^i,\zeta))-\Phi_0(\beta_j^p(x_k^i,\zeta))\big)^+. \qquad (18)$$

$\rightsquigarrow$ Call price. Call option with maturity $T$, strike $K$ on a stock price $X$:

$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$

Considering a self financing portfolio $Y_t$ with $\varphi_t$ assets and bonds with risk free return $r$. We know that the portfolio evolves according to the following dynamics:

$$Y_t = Y_T + \int_t^T f(Y_s, Z_s)ds - \int_t^T Z_s dW_s \tag{19}$$

where the payoff $Y_T = (X_T - K)^+$, the hedging strategy $Z_t = \sigma \varphi_t X_t$ and $f(y, z) = -ry - \frac{\mu - r}{\sigma} z$. It is clear that the function $f$ is linear with respect to $y$ and $z$ and, it is Lipschitz continuous with $[f]_{\mathrm{Lip}} = \max(r, \frac{\mu - r}{\sigma})$. We perform the numerical tests from the algorithm we propose with the following parameters

$$X_0 = 100, \quad r = 0.1, \quad \mu = 0.2, \quad K = 100, \quad T = 0.5$$

and make varying the volatility $\sigma$.

| $\sigma$ | $\widehat{Y}_0$ $(n{=}20)$ | $\widehat{Y}_0$ $(n{=}40)$ | $Y_0$ | $\widehat{Z}_0$ $(n{=}20)$ | $\widehat{Z}_0$ $(n{=}40)$ | $Z_0$ |
|---|---|---|---|---|---|---|
| 0.05 | 04.97 | 05.01 | 05.00 | 04.67 | 04.58 | 04.62 |
| 0.07 | 05.23 | 05.26 | 05.27 | 06.04 | 05.95 | 05.95 |
| 0.10 | 05.81 | 05.84 | 05.85 | 07.83 | 07.72 | 07.71 |
| 0.30 | 10.88 | 10.89 | 10.91 | 19.00 | 18.91 | 19.01 |
| 0.40 | 13.56 | 13.56 | 13.58 | 24.91 | 24.82 | 24.99 |
| 0.50 | 16.26 | 16.25 | 16.26 | 31.07 | 30.98 | 31.24 |

Table: Call price in BS model: $N_k = 100$, $\forall k = 1, \dots, n$; $n \in \{20, 40\}$.

Computational time: $< 1$ second for $n = 20$ and around 1 second for $n = 40$.

⇝ Multidimensional example. We consider the following example due to J.-F. Chassagneux:

$$dX_t = dW_t, \qquad -dY_t = f(t, Y_t, Z_t)dt - Z_t \cdot dW_t$$

where $f(t, y, z) = (z_1 + \ldots + z_d)\big(y - \frac{2+d}{2d}\big)$ and where $W$ is a $d$-dimensional Brownian motion. The solution of this BSDE reads

$$Y_t = \frac{e_t}{1+e_t}, \qquad Z_t = \frac{e_t}{(1+e_t)^2}, \tag{20}$$

with

$$e_t = \exp(x_1 + \ldots + x_d + t).$$

For the numerical experiments, we put the (regular) time discretization mesh to $n = 10$. We choose $t = 0.5$, $d = 2$, so that $Y_0 = 0.5$ and $Z_0^i = 0.24$.

Test for $d = 2$. Using the Markovian product quantization method with $N_1 = N_2 = 30$ we get : $\hat{Y}_0 = 0.504$, $\hat{Z}_0^1 = \hat{Z}_0^2 = 0.2385$. The computation time is around 4 seconds.

## Problems from quantitative finance

1. Several references: *[Pagès, (1998)]*, *[Callegaro, Fiorin, Grasselli (2015)]*, Quantitative Finance: optimal stopping (Bally-Pagès, (2003)/Bally-Pagès-Printems, (2005)), pricing of swing options (Bardou-Bouthemy-Pagès, (2009)), stochastic control (see e.g. Corsi-Pham-Runggaldier (2009) /Pagès-Pham-Printems, (2004)), nonlinear filtering (e.g. Pagès-Pham, (2005) /Pham-Runggal- dier-Sellami, (2005)/etc, variance reduction (Lejay-Reutenauer/Frikha-Sagna., (2012))/ BSDE Illand, Delarue-Menozzi, Chassagneux-Richou, etc, Functional quantization: (Pagès-Printems)