Approximation of Markov semigroup

Advances in Financial Mathematics

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Presentation of the problem - Regular case

- Underlying semigroup :
  $(X_t)_{t\geqslant 0}$ Markov process in $\mathbb{R}^d$.

Let $T>0$, $n\in\mathbb{N}^*$. Time grid : $t_k = k\frac{T}{n}$. $(Z_k)_{k\in\{1,\cdot,n\}}$ sequence of centered and independent random variables in $\mathbb{R}^N$, and $\psi_k \in \mathcal{C}_b^\infty(\mathbb{R}^d\times\mathbb{R}^N\times\mathbb{R}_+)$.

- Approximation semigroup :
  $(\overline{X}^n_{t_k})_{k\in\{0,\cdot,n\}}$ Markov chain : $\overline{X}^n_{t_{k+1}} = \psi_k(\overline{X}^n_{t_k}, \frac{Z_{k+1}}{\sqrt{n}}, T/n)$.
- Weak error, order h for a class of test functions f :

$$|\mathbb{E}[f(X_T)-f(\overline{X}^n_{t_n})]| \leqslant \mathcal{C}(f)/n^h.$$

Convergence for the total variation distance : $\mathcal{C}(f) = C\|f\|_\infty$, for every measurable and bounded function f.

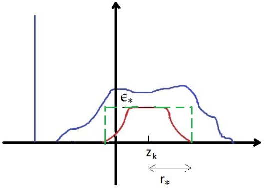
Hypothesis

- Short time approximation : $\Delta_k^n f(x) = \mathbb{E}[f(X_{t_{k+1}}) - f(\overline{X}^n_{t_{k+1}})|X_{t_k} = \overline{X}^n_{t_k} = x]$.

$$\|\Delta_k^n f\|_\infty \leqslant C\|f\|_{q,\infty}/n^{h+1}$$

- **Doeblin** : The law of every Z_k is lower bounded by the Lebesgue measure.

Then, $Z_k = \chi_k U_k + (1-\chi_k)V_k$, χ_k Bernoulli random variable.
Integration by parts formula for functionals of $U_1, .., U_n$.



- Scheme function ψ_k. **Regularity** + **Ellipticity**

Total variation convergence and density approximation - Regular case

Theorem

- $\overline{X}^n$ converges to X for the total variation distance with order h.

$$|\mathbb{E}[f(X_T) - f(\overline{X}^n_{t_n})]| \leqslant C\|f\|_\infty / n^h$$

- Moreover, $\forall t > 0$, $X_t(x)$ has a density $p_t(x,y) \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $\forall R > 0,\ \varepsilon \in (0,1)$

$$\sup_{2S < t_k \leqslant T} \sup_{|x|+|y| \leqslant R} \left| \partial_x^\alpha \partial_y^\beta p_{t_k}(x,y) - \partial_x^\alpha \partial_y^\beta \overline{p}^n_{t_k}(x,y) \right| \leqslant C_\varepsilon / n^{h(1-\varepsilon)}$$

Proof : Finite dimensional Malliavin calculus with respect to U_k, $k \in \{1, .., n\}$. Interpolation inequality for the density approximation. (Bally Caramelino)

Example - Euler

- Euler scheme, $t_k = \frac{kT}{n}$.

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$
$$X^n_{t_{k+1}} = X^n_k + b(X^n_{t_k})(t_{k+1} - t_k) + \sigma(X^n_{t_k})\sqrt{t_{k+1} - t_k} Z_{k+1}$$

  - Z_k Doeblin + Moment hypothesis.
  - $\psi(x, z, t) = x + b(x)t + \sigma(x)z, \quad b, \sigma \in \mathcal{C}^\infty_b(\mathbb{R}), \quad \inf_{x\in\mathbb{R}} \sigma(x) \geqslant \lambda_* > 0.$

  Then,

$$|\mathbb{E}[f(X_T) - f(X^n_T)]| \leqslant \frac{C}{n}\|f\|_\infty$$

Proof Gaussian case : Bally Talay

Total variation convergence - Locally regular case

Theorem

We suppose that there exists $\mathbf{K} \subset \mathbb{R}^d$ such that : $\psi_k \in \mathcal{C}_b^\infty(\mathbf{K} \times \mathbb{R}^N \times \mathbb{R}_+)$, $k \in \{1, .., n\}$, then, for every $\tilde{\mathbf{K}} \subset \mathbf{K}$ (strictly) :

- $\overline{X}_T^n \mathbf{1}_{\tilde{\mathbf{K}}}$ converges to $X_T \mathbf{1}_{\tilde{\mathbf{K}}}$ for the total variation distance.

$$|\mathbb{E}[f(X_T \mathbf{1}_{\tilde{\mathbf{K}}}) - f(\overline{X}_{t_n}^n \mathbf{1}_{\tilde{\mathbf{K}}})]| \leqslant C\|f\|_\infty \varepsilon_\infty(n)/n^h$$

with $\lim_{n\to\infty} \varepsilon_\infty(n) = \infty$.

Proof : Localization of the processes on $\mathbf{K}$ + Concentration inequalities (Berstein, Hoeffding).

Application : Cubature scheme CIR (Alfonsi).

Conclusion

- Weak error for regular and locally regular Brownian diffusion processes.
- Perspectives :
  - Levy processes.
  - Piecewise deterministic Markov processes.
  - General processes with jump (e.g : Processes with censored jumps).

Thank you.