When Almgren Chris meet Black Scholes derivative hedging under illiquid market

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Joint work with O. Guéant. Work supported by HSBC France within the Research Initiative "Modélisation des marchés financiers à haute fréquence" (formerly "Exécution optimale et statistiques de la liquidité haute fréquence")

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Outline

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Introduction

Classical framework: Black-Scholes or Bachelier $\rightarrow$ frictionless market, price-taker agent

- Continuous trading
- Pricing using a risk-neutral expectation
- Replicating portfolio
- Replicating strategy given by the Δ of the option

Liquidity-related questions:

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- large nominal?
- large Γ?
- difference between physical and cash settlement?

Framework

Model:

- Continuous time modelling
- Almgren-Chriss-like market impacts
- Expected utility framework (CARA)
- Indifference pricing
- Partial differential equation (viscosity solutions of HJB)

Features:

- Market impact and execution costs
- Partial hedge
- Modelling Physical delivery

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Call Option

We consider that we have sold this call option with physical settlement.

Characteristics of the Call option

- Strike K
- Maturity T
- Nominal N (in shares)

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N matters because the introduction of execution costs and market impact makes the problem a non-linear one.

Notations

- Let $\left(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space
- The number of shares in the hedging portfolio

$$q_t = q_0 + \int_0^t v_s ds,$$

where q_0 denotes the number of shares in the portfolio at inception

- The process (v) lies in the admissible set:

$$\mathcal{A} = \left\{ v \in \mathcal{P}(0,T), \int_0^T |v_t| dt \in L^\infty(\Omega) \right\}$$

where $\mathcal{P}(s,t)$ is the set of $\mathbb{R}$-valued progressively measurable processes on $[s,t]$

Permanent Market Impact

We could consider the classical linear model:

Linear Permanent Market Impact Model

$$dS_t = \sigma dW_t + k v_t dt$$

Here we consider the more general framework:

General Permanent Market Impact Model

$$dS_t = \sigma dW_t + f\left(|q_0 - q_t|\right) v_t dt$$

where $f : \quad \mathbb{R}_+^* \to \mathbb{R}_+$ nonincreasing and in $L^1_{loc}(\mathbb{R}_+)$.

Instantaneous Market Impact (Execution costs)

Execution Costs Function

Execution costs are modeled by $L \in C(\mathbb{R}, \mathbb{R}_+)$ verifying:

- $L(0) = 0$
- L is an even function
- L is increasing on $\mathbb{R}_+$
- L is strictly convex
- L is asymptotically superlinear: $\lim_{\rho \to +\infty} \frac{L(\rho)}{\rho} = +\infty$

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In practice

L is a function of the following form:

$$L(\rho) = \eta|\rho|^{1+\phi} + \psi|\rho|$$

Instantaneous Market Impact (Execution costs)

Impact on Cash

The Cash account X evolves as:

$$dX_t = -v_t S_t dt - V_t L\left(\frac{v_t}{V_t}\right) dt,\ X_0 = 0$$

where the process $(V_t)_t$ is the market volume process, assumed to be deterministic, positive and bounded.

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Payoff

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- If the option is exercised:

$$\underbrace{X_T}_{\text{cash account}} + \underbrace{KN}_{\text{payment of the client}} - \underbrace{((N - q_T)S_T + \mathcal{L}(q_T, N))}_{\text{cost of buying } (N-q_T) \text{ shares}}$$

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- Otherwise, the payoff is:

$$\underbrace{X_T}_{\text{cash account}} + \underbrace{q_T S_T - \mathcal{L}(q_T, 0)}_{\text{gain of selling the } q_T \text{ shares}}$$

Payoff

The payoff can then be written as (the threshold is assumed to be $K' \leq K$):

Payoff

$$X_T + q_T S_T + 1_{S_T \geq K'} \left(N(K - S_T) - \mathcal{L}(q_T, N)\right) - 1_{S_T < K'} \mathcal{L}(q_T, 0)$$

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To ensure the absence of dynamic arbitrage, we specify $\mathcal{L}$ as:

$$\mathcal{L}(q, q\prime) = \ell(q\prime - q) + q\prime(G(q\prime) - G(q)) - (F(q\prime) - F(q))$$

where ℓ is the cost function when there is no permanent market impact, $F(q) = \int_{q_0}^{q} zf(|q_0 - z|)dz$ and $G(q) = \int_{q_0}^{q} f(|q_0 - z|)dz$.

Outline

Optimization Problem

Optimization Problem

The bank maximizes its expected utility:

$$\sup_{v\in\mathcal{A}} \mathbb{E}\Big[-\exp\Big(-\gamma\Big(X_T + q_T S_T + 1_{S_T \geq K'}\left(N(K - S_T) - \mathcal{L}(q_T, N)\right) - 1_{S_T < K'}\mathcal{L}(q_T, 0)\Big)\Big)\Big]$$

where γ is the absolute risk aversion parameter of the bank.

This framework permits to define a price for the option using indifference pricing.

Value Function

To solve this problem, we introduce the value function u:

Definition

$$u(t,x,q,S) = \sup_{v\in\mathcal{A}_t} \mathbb{E}\Big[-\exp\Big(-\gamma\Big(X_T^{t,x,v} + q_T^{t,q,v} S_T^{t,S,v} + 1_{S_T^{t,S,v}\geq K'}\Big(N(K - S_T^{t,S,v}) - \mathcal{L}(q_T^{t,q,v}, N)\Big) - 1_{S_T^{t,S,v}<K'}\mathcal{L}(q_T^{t,q,v}, 0)\Big)\Big)\Big],$$

where $\mathcal{A}_t = \left\{v \in \mathcal{P}(t,T), \int_t^T |v_s| ds \in L^\infty(\Omega)\right\}$.

HJB Equation

u satisfies the following HJB equation:

HJB equation

$$-\partial_t u - \frac{1}{2}\sigma^2\partial^2_{SS}u - \sup_{v\in\mathbb{R}}\left\{v\partial_q u + \left(-vS - L\left(\frac{v}{V_t}\right)V_t\right)\partial_x u - \partial_S u f(|q_0 - q|)v\right\} = 0$$

with terminal condition:

$$u(T,x,q,S) = -\exp\Big(-\gamma\Big(x + qS - 1_{S<K'}\mathcal{L}(q,0)) + 1_{S\geq K'}\left(N(K-S) - \mathcal{L}(q,N)\right)\Big)\Big)$$

Outline

Change of Variables

We use the following change of variables:

Definition

We introduce θ by:

$$u(t,x,q,S) \;=\; -\exp(-\gamma(x+qS-F(q)-\theta(t,q,S-G(q))))$$

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Indifference Price

$\theta(0, q_0, S_0)$ is the indifference price to write the call option.

PDE for θ

The PDE satisfied by θ is the following:

PDE

$$-\partial_t\theta - \frac{1}{2}\sigma^2\partial^2_{\tilde{S}\tilde{S}}\theta - \frac{1}{2}\gamma\sigma^2(\partial_{\tilde{S}}\theta - q)^2 + V_t H(\partial_q\theta) = 0$$

where H is the Legendre transform of L:

$$H(p) = \sup_\rho \{p\rho - L(\rho)\}$$

Terminal condition:

$$\theta(T, q, \tilde{S}) = 1_{\tilde{S} \geq K' - G(q)} \Big(N(\tilde{S} - K) + NG(N) - F(N) + \ell(N - q) \Big) + 1_{\tilde{S} < K' - G(q)} (\ell(q) - F(0))$$

PDE

Interpretation of the PDE:

$$\underbrace{-\partial_t\theta - \frac{1}{2}\sigma^2\partial^2_{\tilde{S}\tilde{S}}\theta}_{\text{Diffusion}} - \underbrace{\frac{1}{2}\gamma\sigma^2(\partial_{\tilde{S}}\theta - q)^2}_{\text{"Mishedge"}} + \underbrace{V_t H(\partial_q\theta)}_{\text{Execution costs}} \quad = \quad 0$$

An optimal control is given by:

Optimal Control

$$v^\star(t,q,S) = H'(\partial_q\theta(t,q,S-G(q)))$$

Outline

Reference Scenario

- $S_0 = K = K' = 45$ €
- $\sigma = 0.6$ €$\cdot$day$^{-1/2}$ (≈ 21% annual volatility)
- $T = 60$ days
- $V = 1\ 000\ 000$ shares$\cdot$day^{-1}
- $V_d = 1\ 000\ 000$ shares
- $N = 5\ 000\ 000$ shares
- $L(\rho) = \eta|\rho|^{1+\phi}$ with $\eta = 0.1$ € $\cdot$shares$^{-1} \cdot$ day^{-1} and $\phi = 0.75$
- $\gamma = 10^{-6}$€$^{-1}$

Reference Scenario 1 (No PMI)

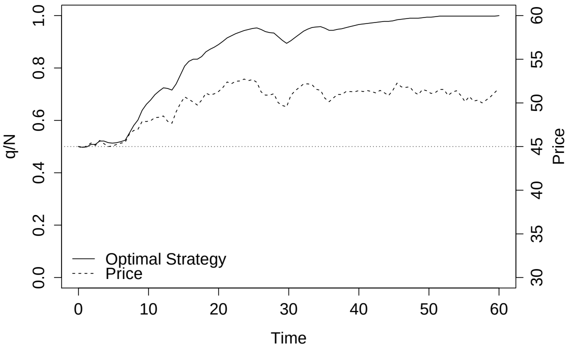

Figure: Reference Scenario 1 - Stock Price and Optimal Strategy

Reference Scenario 2 (No PMI)

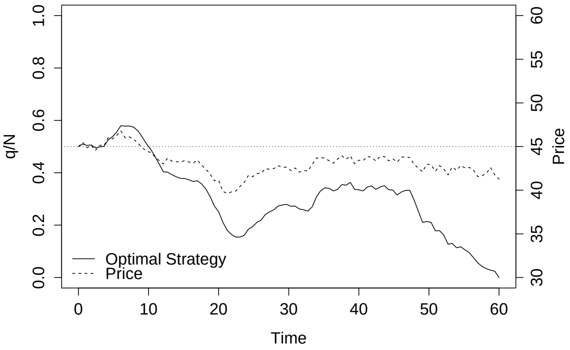

Figure: Reference Scenario 2 - Stock Price and Optimal Strategy

Outline

Importance of Initial Position

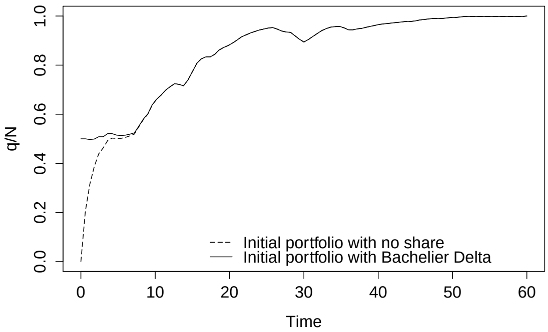

Figure: Optimal portfolio when prices follow trajectory 1

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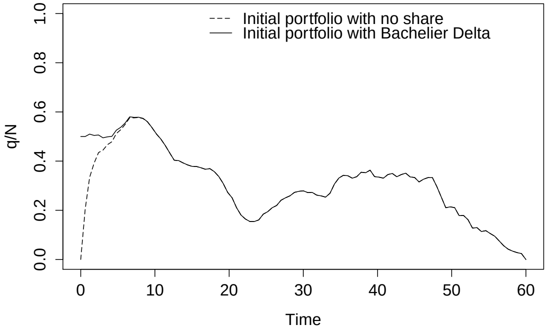


Figure: Optimal portfolio when prices follow trajectory 2

Importance of Initial Position

The prices are given by:

	$q_0 = 0$	$\frac{q_0}{N} = 0.5$
Price of the call	2.54	2.19
Implied σ in the Bachelier model	0.82	0.71

Building the initial position in Δ is usually costly.

Outline

Effect of Permanent Impact

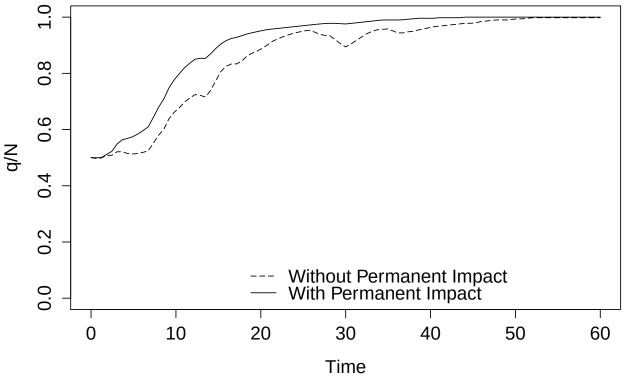
We add the following permanent impact:

Permanent Impact

f is now defined as:

$$f(q) = 0.001 \frac{1}{\sqrt{|q|}}$$

Trajectory 1 - Strategies



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Trajectory 1 - Impacted Prices

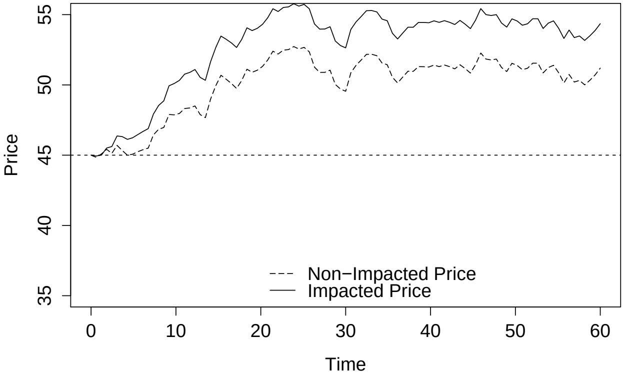

Figure: Prices (trajectory 1) with the influence of permanent market impact

Trajectory 2 - Strategies

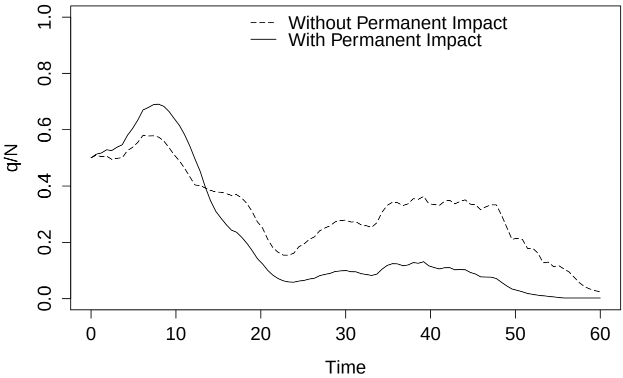

Figure: Optimal portfolio when prices follow trajectory 2

Trajectory 2 - Impacted Prices

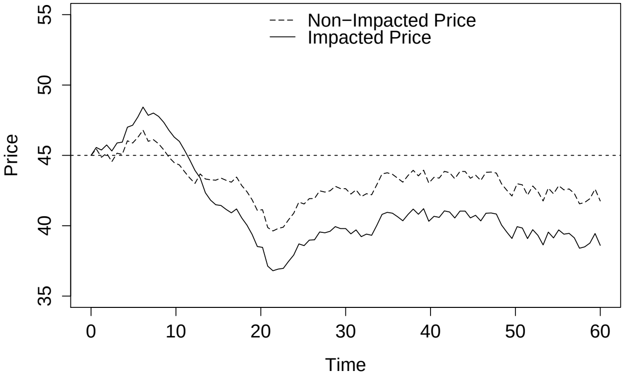

Figure: Prices (trajectory 2) with the influence of permanent market impact

Effect of Permanent Impact

- Mechanical effect: if the price goes up (resp. down), our position goes up (resp. down), and it pushes the price up (resp. down).
- Strategical effect: near maturity and near the money, the bank has incentives to sell shares to push down the price so that the option expires worthless.

Outline

Execution Costs

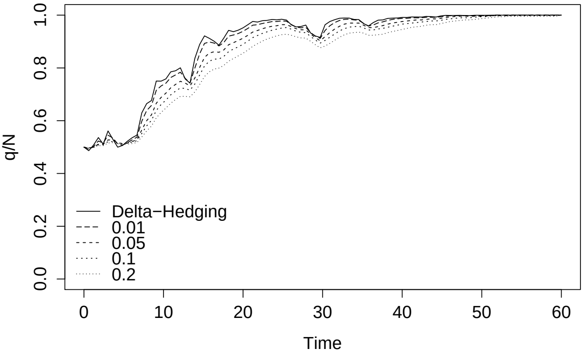

Figure: Optimal portfolio for different values of η, when prices follow Trajectory 1

Execution Costs

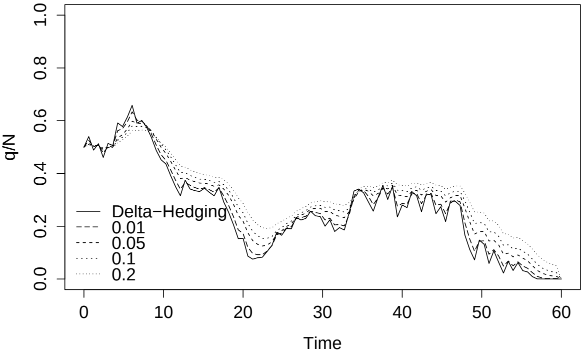

Figure: Optimal portfolio for different values of η, when prices follow Trajectory 2

Execution Costs

When η increases:

- The trajectories are smoother
- They are closer to the position $0.5N$ to avoid round trips

When $\eta \to 0$, we obtain the limiting case of Δ-Hedging.

The prices are given by:

η	0.2	0.1	0.05	0.01	0 (Bachelier)
Price of the call	2.36	2.19	2.07	1.92	1.85
Implied σ in a Bachelier model	0.76	0.71	0.67	0.62	0.6

- Prices are higher when η increases.

Bid-Ask Spread

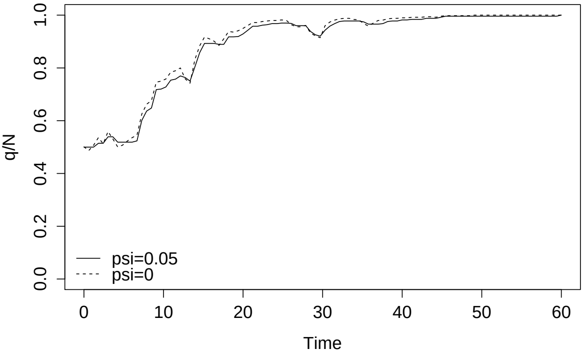

Figure: Optimal portfolio in presence of bid-ask spread, when prices follow Trajectory 1

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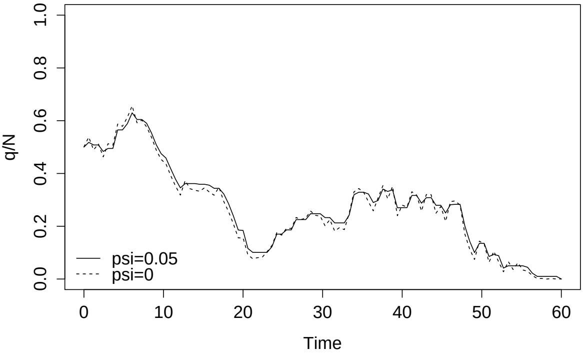

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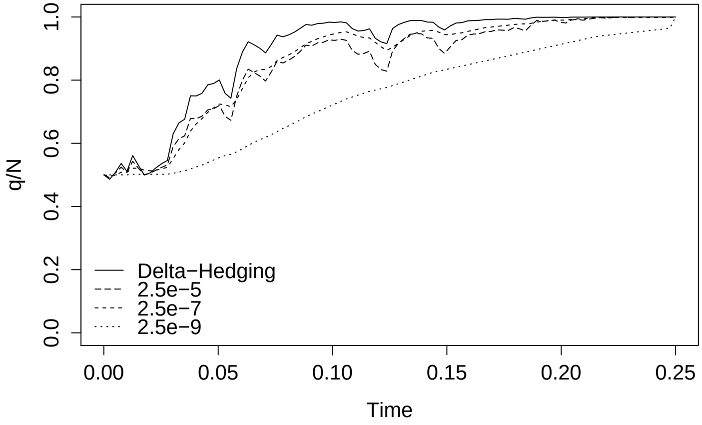
Risk Aversion



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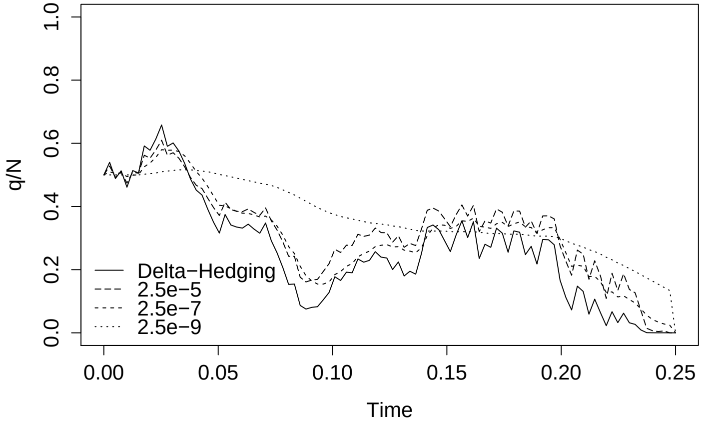

Questions?