Limited liability, or how to prevent slavery in contract theory

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Outline

Motivation

B. Salanié, The economics of contracts

Customers know more about their tastes than firms, firms know more about their costs than the government and all agents take actions that are at least partly unobservable.

- Vast economic literature revisiting general equilibrium theory by incorporating incitations and asymmetry of information.
- Moral hazard: situation where an Agent can benefit from an action (inobservable), whose cost is incurred by others.
- How to design "optimal" contracts?

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Modelisation

- Contract between an Agent and a Principal, between 0 and T.
- Agent chooses his action (or effort): process α.
- Choice of Agent impacts the **distribution** of another process X

$$X_t = X_0 + \int_0^t \alpha_s ds + \sigma W_s^\alpha,\ t \in [0,T],$$

where W^α is a $\mathbb{P}^\alpha$−Brownian motion.

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Agent's problem

- Principal proposes a contract to Agent at 0. This corresponds to a salary/price/premium ξ received at T, contingent on X.
- Agent then solves

$$V_0^A(\xi) := \sup_\alpha \mathbb{E}^{\mathbb{P}^\alpha}\Bigg[\underbrace{U_A}_{\text{Utility}}\bigg(\underbrace{\xi(X)}_{\text{Salary}} - \int_0^T \underbrace{\frac{c}{2}|\alpha_t|^2\,dt}_{\text{Cost}}\bigg)\Bigg].$$

  with $U_A(x) := -\exp(-\gamma_A x)$.
- Dependence of ξ in the whole trajectory of X is, in general, **crucial**.
- Agent faces a non–Markovian stochastic control problem.

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Solving the Agent's problem

- Dynamic version of Agent's utility at time t is

$$V_t^A(\xi) := \underset{\alpha}{\mathrm{esssup}} J_t^\alpha,\ J_t^\alpha = \mathbb{E}^{\mathbb{P}^a}\left[U_A\left(\xi(X) - \int_t^T \frac{c}{2}|\alpha_s|^2 ds\right)\middle|\mathcal{F}_t\right]$$

- Introduce the certainty equivalent $Y := -\log(V^A(\xi))/R_A$
- Itô's formula + classical arguments imply that Y^A solves the BSDE

$$Y_t^A = \xi + \int_t^T f(Z_s)ds - \int_t^T Z_s \sigma dW_s,$$

with $f : z \longmapsto -\frac{\gamma_A}{2}\sigma^2 z^2 + \sup_{a\in\mathbb{R}}\{az - \frac{c}{2}a^2\} = \frac{1}{2}(\frac{1}{c} - \gamma_A\sigma^2)z^2$.

- Optimal effort $\alpha^\star := a^\star(Z_s) := \frac{Z_s}{c}$

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Principal looks for Stackelberg equilibrium in two steps.

(i) Compute best reaction of Agent to a contrat $\xi \longrightarrow \alpha^\star(\xi) \longrightarrow \mathbb{P}^\star(\xi)$.

(ii) Optimisation feedback on the contracts

$$V_0^P := \sup_{\xi \in \Xi_R} \mathbb{E}^{\mathbb{P}^\star(\xi)}\Big[U(X - \xi(X))\Big],$$

- U: utility function of Principal, $U(x) := -\exp(-\gamma_P x)$.
- Ξ_R: contrats such that $V^A(\xi) \geq R$ (participation constraint).

- Direct computation lead to linear optimal contract $\xi^\star := C + z^\star X_T$, with constant effort given by $z^\star/c$ where

$$z^\star := \frac{\gamma_P + \frac{1}{c\sigma^2}}{\gamma_A + \gamma_P + \frac{1}{c\sigma^2}}.$$

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A problem?

- $z^\star$ is deterministic.
- Contract is linear, Markovian, explicit: life is great.
- However, under $\mathbb{P}^\star$, X_T is a drifted BM $\Longrightarrow \mathbb{P}^\star(X_T < 0) > 0$.
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The model

- Two more ingredients compared to before:
  - Agent can only be paid non-negative salary.
  - To make Principal happy, we allow him to fire Agent
- Therefore, contracts are now described by a pair $((\xi_t)_{t\in[0,T]},\tau) \longrightarrow$ salary and firing time
- Limited liability extension of Holmström and Milgrom's model, or finite horizon version of Sannikov's model.

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- Exactly as before, the certainty equivalent of Agent verifies

$$Y_t^A = \xi_\tau + \int_t^\tau f(Z_s)ds - \int_t^\tau Z_s \sigma dW_s,$$

- Clearly, utility of Agent is higher than if he were paid 0 $\Longrightarrow$ comparison theorem.
- Since $f(0) = 0$, certainty equivalent of Agent paid 0 IS 0 (extends to general setup as soon as $c(0) = 0$).
- Therefore

$$Y_t^A \geq 0, \ \forall t \in [0, \tau].$$

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State constraint reinterpretation

- Certainty equivalent Y^A of Agent paid non-negative salary verifies

$$\exists Z \text{ s.t. } Y_t^A = Y_0^A - \int_0^t f(Z_s)ds + \int_0^t Z_s \sigma dW_s, \text{ and } Y_t^A \geq 0.$$

- Converse is true! Any non-negative payment ξ_τ is the terminal value Y_τ^Z of a controlled diffusion as above, constrained to stay positive.

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Principal's problem

- Principal now solves a mixed optimal control/stopping problem with state constraints

$$V^P = \sup_{(\tau,Z)} \mathbb{E}^{\mathbb{P}^\star(Z)}[U(X_T - Y_T^Z)].$$

- Easy to solve the problem on the boundary $y = 0 \longrightarrow$ immediate stopping is optimal (otherwise, optimal stopping problem).
- HJB equation: $u(t,x,y) =: -\exp(-\gamma_P(x - f(t,y))$, with

$$\begin{cases} \max\left\{ -f_t - \frac{\gamma_P\sigma^2}{2} f + \frac{(1+\gamma_P\sigma^2 f_y)^2}{2((\gamma_A\sigma^2+1)f_y + \sigma^2(f_{yy}+\gamma_P f_y^2))^+},\ f - y\right\} = 0, \\ f(t,0) = 0, \\ f(T,y) = y, \end{cases}$$

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Interpretation

- Main findings of Sannikov
  - Agent is not necessarily held to his reservation utility.
  - Agent is fired in two cases: his certainty equivalent reaches 0 (bankruptcy), or it becomes to high (golden parachute)
- In our model, a necessary condition for "golden parachutes" to happen is

$$\gamma_P \sigma^2 (\gamma_A \sigma^2 - 1) \geq 1$$

- Sannikov's result seems to depend heavily on the choice of utility functions. Is it due to exponential utility?

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