Multi-Martingale Optimal Transport

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ADVANCES IN FINANCIAL MATHEMATICS

Martingale Optimal Transport (MOT) Problem in One dimension

- Borel probability measures μ, ν on $\mathbf{R}$ in convex order: $\mu \leq_c \nu$
- (continuous) cost function $c : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$

- $\mathrm{MT}(\mu, \nu)$: probability measures π on $\mathbf{R} \times \mathbf{R}$ which not only project to the marginals μ, ν, but also its disintegration $(\pi_x)_{x \in \mathbf{R}}$ has barycenter at x (martingale constraint):

$$f(x) \leq \int_{\mathbf{R}} f(y)\, d\pi_x(y) \quad \forall f \text{ convex.}$$

- Disintegration = Conditional probability: $\pi_x(A) = \mathbb{P}(Y \in A | X = x)$.

Study the optimal solutions of the minimization problem

$$\min_{\pi \in \mathrm{MT}(\mu,\nu)} \int_{\mathbf{R} \times \mathbf{R}} c(x, y) d\pi(x, y).$$

Probabilistic statement of MOT

- $(\Omega, \mathcal{F}, \mathbb{P})$: probability space
- $X : \Omega \to \mathbf{R}$, $Y : \Omega \to \mathbf{R}$: random variables
- cost function $c : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$
- $\text{Law}(X) = \mu$, $\text{Law}(Y) = \nu$
- $E(Y|X) = X$.

Study the one-step martingales (stocks) (X, Y) with prescribed marginals, which **minimize** the expected cost (option price)

$$\min_{X\sim\mu, Y\sim\nu, E(Y|X)=X} E_{\mathbb{P}} c(X, Y).$$

Motivation:

- [Model-free Finance] find the minimum price of option $c(x, y)$ given market information μ, ν, that is, given the prices of call / put options.

A structure result in 1-dimension

Theorem (Hobson-Neuberger-Klimmek, Beiglböck-Juillet '13)

Let $c(x, y) = \pm|x - y|$ and $d = 1$ (In financial term, this means that the option $|X - Y|$ depends only on one stock process), and assume μ is dispersed ($\mu << \mathcal{L}^1$). Then the optimal martingale transport π is unique for any given ν, and it exhibits an extremal property: for each $x \in \mathbf{R}$, the conditional probability π_x is concentrated at two boundary points of an interval.

Question: What is a right generalization of this theorem in higher dimension?

Multi-Martingale Optimal Transport (MMOT) Problem [L. '16]

- probability measures μ_i, ν_i on $\mathbf{R}$ in convex order, i=1,2,...,d
- cost function (option) $c : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$
- (X_i, Y_i): one-step martingales ($\mathbb{E}(Y_i|X_i) = X_i$) with the prescribed marginal laws $X_i \sim \mu_i$ and $Y_i \sim \nu_i$
- $\mu := (\mu_1, ..., \mu_d), \quad \nu := (\nu_1, ..., \nu_d)$
- $\mathrm{MMT}(\mu, \nu)$: the set of probability measures on $\mathbf{R}^d \times \mathbf{R}^d$ such that each $\pi \in \mathrm{MMT}(\mu, \nu)$ is the joint law of martingales $(X_i, Y_i)_{i \le d}$ having $(\mu_i, \nu_i)_{i \le d}$ as its marginals, respectively.

Study the optimal solutions of the minimization problem

$$\text{Minimize} \quad \mathrm{Cost}[\pi] = \int_{\mathbf{R}^d \times \mathbf{R}^d} c(x, y)\, d\pi(x, y) \quad \text{over} \quad \pi \in \mathrm{MMT}(\mu, \nu).$$

Motivation:

- [Finance] find the minimum price of the option whose value depends on many stocks (X_i, Y_i), $i = 1, ..., d$, given the information that can be observed from the market.

Probabilistic description of MMOT

- $(\Omega, \mathcal{F}, \mathbb{P})$: probability space
- $X_i : \Omega \to \mathbf{R}$, $Y_i : \Omega \to \mathbf{R}$: random variables, $i = 1, 2, ..., d$
- cost function $c : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$
- $\text{Law}(X_i) = \mu_i$, $\text{Law}(Y_i) = \nu_i$
- $E(Y|X) = X$, where $X = (X_1, ..., X_d)$, $Y = (Y_1, ..., Y_d)$

Study the one-step martingales (stocks) (X, Y) with prescribed marginals, which **minimize** the expected cost (option price)

$$\min_{X_i \sim \mu_i, Y_i \sim \nu_i, E(Y|X) = X} E_{\mathbb{P}} c(X, Y).$$

Extremal structure of MMOT holds true in every dimension

Theorem [L. '16] **Assume:**

- $\mu_i \leq_c \nu_i$ (not necessarily irreducible)
- $\mu_i << \mathcal{L}^1$
- $c(x, y) = \pm||x - y||$ where $|| \cdot ||$ is any strictly convex norm on $\mathbf{R}^d$
- $\pi = \mathrm{Law}(X, Y)$ is any minimizer of MMOT with copula $\pi^1 = \mathrm{Law}(X)$

Then: for any disintegration $(\pi_x)_x$ of π with respect to π^1, the support of π_x coincides with the extreme points of the closed convex hull of itself:

$$\mathrm{supp}\, \pi_x = \mathrm{Ext}\left(\overline{\mathrm{conv}}(\mathrm{supp}\, \pi_x)\right), \quad \pi^1 - a.e.\ x.$$

- Literature in OT:
  Sudakov, Evans, Gangbo, McCann, Ambrosio, Kirchheim, Pratelli, Caffarelli, Feldman, Otto, Kinderlehrer, Jordan, Bianchini, Cavalletti, Ma, Trudinger, Wang, Champion, De Pascale, and others...

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How to obtain such structure result? Study the Dual Optimizer of MOT

- We say that a triple of functions (ϕ, ψ, h) is a dual maximizer of the MOT problem, if for every minimizer π of MOT we have

$$\phi(x) + \psi(y) + h(x) \cdot (y - x) \leq c(x, y) \quad \forall x \in \mathbf{R}, \forall y \in \mathbf{R}, \tag{0.1}$$
$$\phi(x) + \psi(y) + h(x) \cdot (y - x) = c(x, y) \quad \pi - a.e.\,(x, y). \tag{0.2}$$

- $\phi(x) + \psi(y) + h(x) \cdot (y - x)$ can be interpreted as an optimal subhedging strategy for the option $c(x, y)$.

Irreducibility of (μ, ν) is essential to achieve duality in MOT

- Beiglböck-Juillet, Beiglböck-Nutz-Touzi showed that in dimension one ($d = 1$), duality is attained if the marginals (μ, ν) are **irreducible**.

- The irreducibility of (μ, ν) is characterized by their *potential functions*

$$u_\mu(x) := \int |x - y| \, d\mu(y), \quad u_\nu(x) := \int |x - y| \, d\nu(y).$$

- This is also where the OT and MOT are divergent: in OT theory essentially no relation between μ, ν is required for duality.

- The seemingly harmless linear term $h(x) \cdot (y - x)$ drastically changes the picture.

Duality in MMOT (is also possible!)

Theorem [L. '16] **Assume:**

- (μ_i, ν_i) is irreducible, $\forall i = 1, ..., d$
- π is any minimizer of MMOT

Then: there exist a bunch of functions $\phi_i, \psi_i : \mathbf{R} \to \mathbf{R}$, i=1,...,d, $h : \mathbf{R}^d \to \mathbf{R}^d$ which is a dual maximizer:

$$\sum_{i=1}^{d} \phi_i(x_i) + \sum_{i=1}^{d} \psi_i(y_i) + h(x) \cdot (y - x) \leq c(x, y) \quad \forall x \in \mathbf{R}^d, \forall y \in \mathbf{R}^d,$$

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Theorem [L. '16] **Assume:**

- $(\phi_i, \psi_i, h_i)_{i \le d}$ is a dual maximizer
- $\pi = \mathrm{Law}(X, Y)$ is any minimizer of MMOT

Then: its first and second copulas π^1, π^2 (i.e. $\pi^1 = \mathrm{Law}(X)$, $\pi^2 = \mathrm{Law}(Y)$) solve the dual optimal transport problem with respect to the costs α, β respectively:

$$\sum_i \phi_i(x_i) \le \alpha(x) \quad \mu_i - a.e.\ x_i \quad \forall i \in (d), \quad \text{and} \quad \sum_i \phi_i(x_i) = \alpha(x) \quad \pi^1 - a.e.\ x,$$

$$\sum_i \psi_i(y_i) \ge \beta(y) \quad \nu_i - a.e.\ y_i \quad \forall i \in (d), \quad \text{and} \quad \sum_i \psi_i(y_i) = \beta(y) \quad \pi^2 - a.e.\ y.$$

- Here the functions $\alpha : \mathbf{R}^d \to \mathbf{R}$, $\beta : \mathbf{R}^d \to \mathbf{R}$ are naturally defined in terms of the function $y \mapsto \sum_{i=1}^d \psi_i(y_i)$ and are called the martingale Legendre transform. (**Ghoussoub-Kim-L. '15**)

- OT theory can enter for the study of the structure of $\mathrm{Law}(X)$, $\mathrm{Law}(Y)$.

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Conclusion:

- **The duality attainment results presented so far shall serve as the cornerstones for further development of the MOT / MMOT theory, as it did so in the classical OT theory.**

- **As the classical optimal transport theory (in higher dimensions) has made important contributions to many areas of mathematics and economics, I believe that this new development of probabilistic optimal embedding theory in higher dimensions will have far-reaching consequences as well.**

Thank You Very Much!