Large deviations for affine processes and application to variance reduction for pricing

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Joint work with, Z. Grbac and P. Tankov

January 11, 2017

Overview

Presentation of the problem

Denote $(S_u)_{0 \leq u \leq t}$ an underlying and $P(S)$ the payoff of a derivative on S. The price of a derivative is generally calculated as the expectation $\mathbb{E}(P(S))$ under a certain risk-neutral measure $\mathbb{P}$.

We write $S_u = S_0 e^{X_u}$, where we model X as an affine stochastic volatility model [Keller-Ressel, 2011].

Definition and properties of the model

Definition: An affine stochastic volatility model $(X_s, V_s)_{s\leq t}$, is a stochastically continuous, time-homogeneous Markov process such that $\left(e^{X_s}\right)_{s\leq t}$ is a martingale and

$$\mathbb{E}\left(e^{uX_s+wV_s}\middle|X_0=x, V_0=v\right)=e^{\phi(s,u,w)+\psi(s,u,w)\,v+u\,x}\,,$$

for all $(s,u,w)\in\mathbb{R}_+\times\mathbb{C}^2$.

One of the main properties of affine stochastic volatility models is that the functions ϕ and ψ satisfy generalized Riccati equations

$$\begin{aligned}\partial_t\phi(t,u,w)&=F(u,\psi(t,u,w))\,,\quad \phi(0,u,w)=0\\ \partial_t\psi(t,u,w)&=R(u,\psi(t,u,w))\,,\quad \psi(0,u,w)=w\,.\end{aligned}$$

where F and R have Lévy-Khintchine form.

Definition and properties of the model

Theorem Under appropriate hypotheses,

- There exists an interval $I \supseteq [0, 1]$, such that for each $u \in I$, the Generalized Riccati equations admit a unique stable equilibrium $w(u)$ and at most one other equilibrium $\tilde{w}(u)$, which is unstable.
- For $u \in \mathbb{R} \backslash I$, the Generalized Riccati equations do not have any equilibrium.

We denote $\mathcal{B}(u)$ the basin of attraction of the stable solution $w(u)$.

Definition and properties of the model

Denoting $J = \{u \in I \,:\, F(u, w(u)) < \infty\}$. We have that

- J is an interval such that $[0,1] \subseteq J \subseteq I$.
- For $u \in I$, $w \in \mathcal{B}(u)$ and $\Delta t > 0$, we have

$$\psi\left(\frac{\Delta t}{\epsilon},\, u,\, w\right) \underset{\epsilon \to 0}{\longrightarrow} w(u)\,.$$

- For $u \in J$, $w \in \mathcal{B}(u)$ and $\Delta t > 0$,

$$\epsilon\,\phi\left(\frac{\Delta t}{\epsilon},\, u,\, w\right) \underset{\epsilon \to 0}{\longrightarrow} \Delta t\, h\,(u)\ ,$$

  where $h(u) = F(u, w(u))$.

A result of large deviations

Theorem Under some (strong but verifiable) hypotheses on the function h, $(X^{\epsilon}_s)_{0\leq s\leq t} := (\epsilon X_{s/\epsilon})_{0\leq s\leq t}$ satisfies a LDP, as ϵ tends to 0, on $\{x : [0,t] \to \mathbb{R} \,:\, x_0 = 0\}$ equipped with the topology of point-convergence with good rate function

$$\Lambda^*(x) = \int_0^t h^*(\dot{x}^{ac}_s)\, ds + \int_0^t \mathcal{H}\left(\frac{d\nu_s}{d\theta_s}\right) d\theta_s \,,$$

where

$$h^*(y) = \lim_{\epsilon\to 0} \sup_{\lambda\in J} \{\lambda y - h(\lambda)\} \,,$$

$$\mathcal{H}(y) = \lim_{\epsilon\to 0} \epsilon\, h^*(y/\epsilon) = y\left(\mathbb{1}_{\{y>0\}} \sup\{u\in J\} + \mathbb{1}_{\{y<0\}} \inf\{u\in J\}\right) \,,$$

$\dot{x}^{ac}$ is the derivative of the absolutely continuous part of x, ν_s is the singular component of dx_s with respect to ds and θ_s is any non-negative, finite, regular, $\mathbb{R}$-valued Borel measure, with respect to which ν_s is absolutely continuous.

A result of large deviations

Idea of the proof

1. Prove that for $0 < t_1 \leq ... \leq t_n \leq t$, $(X^{\epsilon}_{t_1}, ..., X^{\epsilon}_{t_n})$ satisfies a LDP.
   - Use iteratively the tower property and the expression of the Laplace transform of (X_s, V_s) to obtain the exact expression for $\mathbb{E}\left[e^{\sum_{j=1}^{n} \lambda_j X_{t_j/\epsilon}}\right]$.
   - Use the behaviour of the solutions ϕ and ψ of the Generalized Riccati equations when time tends to infinity to calculate $\lim_{\epsilon \to 0} \epsilon \log\left(\mathbb{E}\left[e^{\sum_{j=1}^{n} \lambda_j X_{t_j/\epsilon}}\right]\right)$.
   - Use the Gärtner-Ellis Theorem.
2. Use the Dawson-Gärtner theorem to extend the LDP to the whole trajectory of $(X^{\epsilon}_s)_{s \leq t}$.
3. Use a convex analysis result by [Rockafellar, 1971] to obtain the form of the rate function.

Monte-Carlo estimation

When the payoff P is too complex to allow to calculate $\mathbb{E}(P(S))$, one often uses Monte-Carlo methods that consists in simulating n independent trajectories $S^{(i)}$ and using the estimator

$$\mathbb{E}(P(S)) \approx \frac{1}{n}\sum_{i=1}^{n} P(S^{(i)}) \,.$$

The estimator is unbiased and has variance

$$\mathbb{V}\mathrm{ar}\left(\frac{1}{n}\sum_{i=1}^{n} P(S^{(i)})\right) = \frac{\mathbb{V}\mathrm{ar}(P(S))}{n} = \frac{1}{n}\left(\mathbb{E}(P^2(S)) - \mathbb{E}^2(P(S))\right) \,.$$

Monte-Carlo estimation with measure change

Let $\mathbb{Q}$ be a measure equivalent to $\mathbb{P}$. Provided we can simulate S under the measure $\mathbb{Q}$, another alternative to estimate $\mathbb{E}(P(S))$ is to simulate n independent trajectories $S^{(i,\mathbb{Q})}$ and use the estimator

$$\mathbb{E}(P(S)) = \mathbb{E}^{\mathbb{Q}}\left(P(S)\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}(S)\right) \approx \frac{1}{n}\sum_{i=1}^{n} P(S^{(i,\mathbb{Q})})\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}(S^{(i,\mathbb{Q})})\,.$$

The new estimator is also unbiased but its variance is

$$\begin{aligned}\mathbb{V}\mathrm{ar}\Big(\frac{1}{n}\sum_{i=1}^{n} P(S^{(i,\mathbb{Q})})\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}(S^{(i,\mathbb{Q})})\Big) &= \frac{\mathbb{V}\mathrm{ar}^{\mathbb{Q}}\left(P(S)\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}(S)\right)}{n} \\ &= \frac{1}{n}\left(\mathbb{E}\Big(P^2(S)\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}(S)\Big) - \mathbb{E}^2(P(S))\right)\,.\end{aligned}$$

Optimal measure change

We consider the class of measures $\mathbb{P}_\theta$ given by time dependent Esscher transform

$$\frac{\mathrm{d}\mathbb{P}_\theta}{\mathrm{d}\mathbb{P}} = \frac{e^{\int_0^t X_s\, d\theta_s}}{\mathbb{E}\left[e^{\int_0^t X_s\, d\theta_s}\right]},$$

where θ is a finite signed measure on $[0, t]$. Denoting $H(X) = \log P(e^X)$, the variance minimization problem writes

$$\inf_\theta \mathbb{E}\left[\exp\left(2H(X) - \int_0^t X_s\, d\theta_s + \log \mathbb{E}\left[e^{\int_0^t X_s\, d\theta_s}\right]\right)\right].$$

Unfortunately, we cannot solve this problem.

Asymptotic minimization problem

We therefore use the LDP of X and Varandhan's lemma to obtain a proxy of the minimization problem

$$\inf_{\theta} \sup_{x} \left\{ 2H(x) - \int_0^t x_s d\theta_s - \Lambda^*(x) \right\} + \int_0^t h(\theta([s,t]))\, ds \,,$$

before using, when H is concave, a result of [Genin and Tankov, 2016], which states that

$$\inf_{\theta} \sup_{x} \left\{ 2H(x) - \int_0^t x_s d\theta_s - \Lambda^*(x) \right\} + \int_0^t h(\theta([s,t]))\, ds = 2 \inf_{\theta} \left\{ \hat{H}(\theta) + \int_0^t h(\theta([s,t]))\, ds \right\} \,,$$

where

$$\hat{H}(\theta) = \sup_{x} \left\{ H(x) - \int_0^t x_s d\theta_s \right\} \,.$$

The case of the European put option

For the European put, we have $H(x) = \log(K - S_0 e^{x_t})_+$, for $\hat{H}(\theta)$ to be finite, the measure θ needs to be supported on $\{t\}$. We therefore denote abusively θ, the value $\theta(\{t\})$. In this case,

$$\hat{H}(\theta) = \log\left(\frac{K}{1-\theta}\right) - \theta \log\left(\frac{-\theta\, K/S_0}{1-\theta}\right) .$$

Therefore, the optimal θ for the European put option is given by

$$\underset{\theta \in \mathbb{R}_-}{\operatorname{argmin}} \log\left(\frac{K}{1-\theta}\right) - \theta \log\left(\frac{-\theta\, K/S_0}{1-\theta}\right) + t\, h(\theta) .$$

The Heston model

We are considering the Heston model

$$dX_s = -\frac{V_s}{2}\,ds + \sqrt{V_s}\,dW_s^1\,, \qquad X_0 = 0$$
$$dV_s = \lambda(\mu - V_s)\,ds + \zeta\sqrt{V_s}\,dW_s^2\,, \qquad V_0 = 1.3$$
$$d\left\langle W^1, W^2\right\rangle_s = \rho\,ds\,,$$

where W^1, W^2 are standard Brownian motions under the measure $\mathbb{P}$, with parameters $\lambda = 1.1$, $\mu = 0.7$, $\zeta = 0.3$, $\rho = -0.5$. In this case,

$$h(u) = \mu\frac{\lambda}{\zeta}\left(\frac{\lambda}{\zeta} - \rho u\right) - \mu\frac{\lambda}{\zeta}\sqrt{\left(\frac{\lambda}{\zeta} - \rho u\right)^2 + \frac{1}{4} - \left(u - \frac{1}{2}\right)^2}$$

and we can obtain the optimal θ for the European put option numerically.

Numerical results

We test our method by simulating, under both $\mathbb{P}$ and $\mathbb{P}_\theta$, 10000 price trajectories $S_s = e^{X_s}$, with 200 discretization steps. We obtain the following results.

t	0.25	0.5	1	2	3
Variance ratio	4.18	3.59	2.95	2.42	2.04

Table : The variance ratio as function of the maturity for a European put option with strike $K = 1$.

K	0.25	0.5	0.75	1	1.25	1.5	1.75
Variance ratio, $t = 1$	8.07	4.49	3.46	3.05	2.69	2.55	2.35
Variance ratio, $t = 3$	3.34	2.57	2.16	2.09	1.93	1.86	1.76

Table : The variance ratio as function of the strike for European put options with maturities $t = 1$ and $t = 3$.

Numerical result for the Asian put option

For the Asian put option the log-payoff function is

$$H(x) = \log \left(K - \frac{S_0}{n} \sum_{j=1}^{n} e^{x_{t_j}} \right)_+ .$$

This makes finding the optimal measure slightly more difficult, but the problem remains tractable. We obtain for the Asian option the following results.

K	0.2	0.4	0.6	0.8	1.0	1.2	1.5	2.0
Variance ratio	29.8	8.96	6.14	4.58	3.90	3.41	2.97	2.74

Table : The variance ratio as function of the strike for the Asian put option with maturity $t = 1.5$.

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Thank you for attending this presentation!