Existence for a calibrated Regime Switching Local Volatility model

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Outline

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Trying to match marginals

- The market gives the prices of European Calls $C(T,K)$ for $T,K>0$ (idealized situation; in practice only $(C(T_i,K_i))_{1\leq i\leq I}$).
- A model $(S_t)_{t\geq 0}$ is calibrated to European options if

$$\forall T,K\geq 0,\ \ C(T,K)=\mathbb{E}\left[e^{-rT}\left(S_T-K\right)^+\right].$$

- By Breeden and Litzenberger (1978), {prices of European Call options for all $T,K>0$} $\iff$ {marginal distributions of $(S_t)_{t\geq 0}$}.
- Dupire Local Volatility model (1992), matching market marginals:

$$dS_t=rS_t dt+\sigma_{Dup}(t,S_t)S_t dW_t$$

$$\sigma_{Dup}(T,K)=\sqrt{2\frac{\partial_T C(T,K)+rK\partial_K C(T,K)}{K^2\partial^2_{KK}C(T,K)}}$$

LSV models

- Motivation: get processes with richer dynamics (e.g. take into account volatility risk) and satisfying marginal constraints.

- Lipton (2002) and Piterbarg (2007),...: Local and Stochastic Volatility (LSV) model

$$dS_t = rS_t + f(Y_t)\sigma(t, S_t)S_t dW_t$$

- "Adding uncertainty" to LV models by a random multiplicative factor $f(Y_t)$ where $(Y_t)_{t\geq 0}$ is a stochastic process and $f > 0$.

Calibration of LSV Models

- By Gyongy's theorem (1988), the LSV model is calibrated to $C(T,K), \forall T, K > 0$ if

$$\mathbb{E}\left[(f(Y_t)\sigma(t,S_t)S_t)^2 | S_t\right] = (\sigma_{Dup}(t,S_t)S_t)^2$$

$$\sigma(t,x) = \frac{\sigma_{Dup}(t,x)}{\sqrt{\mathbb{E}\left[f^2(Y_t)|S_t = x\right]}}$$

- The obtained SDE is nonlinear in the sense of McKean:

$$dS_t = rS_t dt + \frac{f(Y_t)}{\sqrt{\mathbb{E}[f^2(Y_t)|S_t]}}\sigma_{Dup}(t,S_t)S_t dW_t.$$

Simulation results

- Ren, Madan and Qian (2007): solve numerically the associated Fokker-Planck PDE, and get the joint-law of (S_t, Y_t).
- Guyon and Henry-Labordère (2011): efficient calibration procedure based on kernel approximation of the conditional expectation.
  Subsequent extension to stochastic interest rates, stochastic dividends, multidimensional local correlation models,...
- However, calibration errors seem to appear when the range of $f(Y)$ is too large.

Theoretical results

- Abergel and Tachet (2010): perturbation of the constant f case (Dupire) $\longrightarrow$ existence for the restriction to a compact spatial domain of the associated Fokker-Planck equation when $\sup f - \inf f$ small.
- Global existence and uniquess to LSV models remain on open problem.

Outline

A simpler SDE

- The stochastic factor Y is a r.v. (does not evolve with time) with values in $\mathcal{Y} := \{y_1, ..., y_d\}$ such that

$$\forall i \in \{1, ..., d\}, \ \alpha_i = \mathbb{P}(Y = y_i) > 0.$$

It is independent from S_0 and $(W_t)_{t\geq 0}$.

- We suppose $\sigma_{Dup} \equiv 1$, $r = \frac{1}{2}$ and consider $X_t = \ln(S_t)$.
- Since $\mathbb{E}\left[f^2(Y)|S_t\right] = \mathbb{E}\left[f^2(Y)|X_t\right]$ and neglecting the drift term $\frac{1}{2} - \frac{1}{2}\frac{f^2(Y)}{\mathbb{E}[f^2(Y)|X_t]}$ with vanishing conditional expectation given X_t, we get

$$(FBM) \begin{cases} dX_t = \dfrac{f(Y)}{\sqrt{\mathbb{E}[f^2(Y)|X_t]}} dW_t \\ X_0 \sim \mu_0. \end{cases}$$

with X_0, Y and $(W_t)_{t\geq 0}$ independent.

- If $X_0 = 0$, we expect that $X_t \sim \mathcal{N}_1(0, t)$.

Fake Brownian motion

Definition

A fake Brownian motion is a martingale $(M_t)_{t\geq 0}$ such that for all $t \geq 0$, $M_t \sim \mathcal{N}_1(0, t)$ and $(M_t)_{t\geq 0}$ is not a Brownian motion.

Discontinuous fake Brownian motions : Madan and Yor (2002), Hamza and Klebaner (2007), Henry-Labordère, Tan and Touzi (2016),...

Continuous fake Brownian motions : Albin (2010), Oleszkiewicz (2010), Baker, Donati-Martin and Yor (2011),...

Fake exponential Brownian motion : Hobson (2013)

A new continuous fake Brownian motion

Lemma

If the positive function f is not constant on $\mathcal{Y}$, then any solution to the SDE

$$dX_t = \frac{f(Y)}{\sqrt{\mathbb{E}\left[f^2(Y)|X_t\right]}} dW_t, \ X_0 = 0$$

with Y and $(W_t)_{t\geq 0}$ indep. is a continuous fake Brownian motion.

- By Gyongy's theorem, $\forall t \geq 0$, $X_t \sim \mathcal{N}_1(0, t)$.
- If $(X_t)_{t\geq 0}$ is a Brownian motion then a.s. $\forall t \geq 0$, $< X >_t = t$ i.e. ds a.e. $\frac{f^2(Y)}{\mathbb{E}[f^2(Y)|X_s]} = 1 = \frac{f(Y)}{\sqrt{\mathbb{E}[f^2(Y)|X_s]}}$ so that a.s. $\forall t \geq 0$, $X_t = W_t$. Therefore $X_t \perp Y$, $\mathbb{E}\left[f^2(Y)|X_t\right] = \mathbb{E}\left[f^2(Y)\right]$ and $f^2(Y) = \mathbb{E}\left[f^2(Y)\right]$ is constant.

Existence to SDE (FBM) and fake Brownian motion

We define for $i \in \{1, ..., d\}$, $\lambda_i := f^2(y_i)$.

Theorem

Condition (C) : *there exists a symmetric positive definite* $\Gamma \in \mathbb{R}^{d\times d}$ *such that for all* $k \in \{1, \dots, d\}$, *the* $d \times d$ *matrix*

$$\Gamma^{(k)}_{ij} = \frac{\lambda_i + \lambda_j}{2}(\Gamma_{ij} + \Gamma_{kk} - \Gamma_{ik} - \Gamma_{jk}) \text{ is positive definite on } e_k^\perp.$$

Under (C), there exists a weak solution to the SDE (FBM).

If $d = 2$, (C) is satisfied : choice $\Gamma = I_2$,

if $d = 3$, $(C) \Leftrightarrow \frac{1}{\beta_1\beta_2} + \frac{1}{\beta_2\beta_3} + \frac{1}{\beta_3\beta_1} > \frac{1}{4}$ with

$$\beta_1 = \left|\sqrt{\frac{\lambda_2}{\lambda_3}} - \sqrt{\frac{\lambda_3}{\lambda_2}}\right|, \beta_2 = \left|\sqrt{\frac{\lambda_3}{\lambda_1}} - \sqrt{\frac{\lambda_1}{\lambda_3}}\right|, \beta_3 = \left|\sqrt{\frac{\lambda_1}{\lambda_2}} - \sqrt{\frac{\lambda_2}{\lambda_1}}\right|$$

if $d \geq 4$, $\max_{1\leq k\leq d} \sum_{i\neq k} \lambda_i \sum_{i\neq k} \frac{1}{\lambda_i} \leq (d+1)^2 \Rightarrow (C)$: $\Gamma = I_d$.

Where does (C) appear?

The associated Fokker Planck system

- For $i \in \{1, ..., d\}$, define p_i s.t., for $\phi \geq 0$ and measurable, $\mathbb{E}\left[\phi\left(X_t\right) 1_{\{Y=y_i\}}\right] = \int_{\mathbb{R}} \phi(x) p_i(t,x) dx$.

- The associated Fokker-Planck system is:

$$\forall i \in \{1, ..., d\}, \partial_t p_i = \frac{1}{2}\partial^2_{xx}\left(\frac{\sum_j p_j}{\sum_j \lambda_j p_j}\lambda_i p_i\right)$$
$$p_i(0) = \alpha_i \mu_0$$

- $\sum_j p_j$ is solution to the heat equation.

Rewriting into divergence form

The system satisfied by $p = (p_1, \dots, p_d)$ can be rewritten in divergence form:

$$\partial_t p = \frac{1}{2}\partial_x \left(\sum_{k=1}^{d} w_k(p) M^{(k)}(p) \partial_x p \right)$$

where $w_k(p) := \frac{\lambda_k p_k}{\sum_j \lambda_j p_j}$, $\sum_{k=1}^{d} w_k(p) = 1$ and for $\overline{\lambda}(p) := \frac{\sum_j \lambda_j p_j}{\sum_j p_j}$,

$$M^k(p) := \begin{pmatrix} \frac{\lambda_1}{\overline{\lambda}} & & & & & & \\ & \cdot & & & & & \\ & & \frac{\lambda_{k-1}}{\overline{\lambda}} & & & & \\ \left(1 - \frac{\lambda_1}{\overline{\lambda}}\right) & \cdot & \left(1 - \frac{\lambda_{k-1}}{\overline{\lambda}}\right) & 1 & \left(1 - \frac{\lambda_{k+1}}{\overline{\lambda}}\right) & \cdot & \left(1 - \frac{\lambda_d}{\overline{\lambda}}\right) \\ & & & & \frac{\lambda_{k+1}}{\overline{\lambda}} & & \\ & & & & & \cdot & \\ & & & & & & \frac{\lambda_d}{\overline{\lambda}} \end{pmatrix} \leftarrow \text{row } k$$

Computing Energy Estimates

- Multiply the system by $p^*(J_d+\epsilon\Gamma)$ where $(J_d)_{ij}=1$ and $\epsilon>0$, and integrate in $x\in\mathbb{R}$ by parts :

$$\frac{1}{2}\partial_t\left(\int_{\mathbb{R}} p^*(J_d+\epsilon\Gamma)p dx\right) = -\frac{1}{2}\int_{\mathbb{R}}\sum_{k=1}^{d} w_k(p)\partial_x p^*(J_d+\epsilon\Gamma)M^{(k)}(p)\partial_x p dx.$$

- By (C), $\exists\eta>0$, $\forall y=(y_1,\dots,y_d)\in\mathbb{R}^d$ with $\sum_{j=1}^d y_j=0$,

$$\bar{\lambda}(p)y^*\Gamma M^{(k)}(p)y=(y-y_ke_k)^*\Gamma^{(k)}(y-y_ke_k)\geq\eta|y|^2.$$

- Since $J_dM^{(k)}(p)=J_d$ so that $y^*J_dM^{(k)}(p)y=(\sum_{i=1}^d y_i)^2$, we deduce that for ϵ small enough,

$$\inf_{p,y\in\mathbb{R}^d}\frac{1}{|y|^2}\sum_{k=1}^{d} w_k(p)y^*(J_d+\epsilon\Gamma)M^{(k)}(p)y>0.$$

- $\Rightarrow$ EE in $L^2([0,T],(H^1(\mathbb{R}))^d)\cap L^\infty([0,T],(L^2(\mathbb{R}))^d)$.

Step 1/3: Existence to an approximate PDS when $\mu_0(dx) = p_0(x)dx$, $p_0 \in L^2(\mathbb{R})$

For $\epsilon > 0$, use Galerkin's method to solve an approximate PDE:

$$\partial_t p^\epsilon = \frac{1}{2}\partial_x \left(M^\epsilon(p^\epsilon)\partial_x p^\epsilon \right) \quad p^\epsilon(0) = (\alpha_1, \dots, \alpha_d)\, p_0.$$

where
$$\sum_{k=1}^{d} w_k(\rho) M_{ii}^{(k)}(\rho) = 1 + \frac{\sum_{l \neq i} \lambda_l \rho_l \sum_l (\lambda_i - \lambda_l)\rho_l}{\left(\sum_l \lambda_l \rho_l\right)^2}$$
$$\longrightarrow M_{ii}^\epsilon(\rho) = 1 + \frac{\sum_{l \neq i} \lambda_l \rho_l^+ \sum_l (\lambda_i - \lambda_l)\rho_l^+}{\left(\epsilon \vee \sum_l \lambda_l \rho_l^+\right)^2},$$

and for $j \neq i$,
$$\sum_{k=1}^{d} w_k(\rho) M^{(k)}(\rho)_{ij} = \frac{\lambda_i \rho_i \sum_l (\lambda_l - \lambda_j)\rho_l}{\left(\sum_l \lambda_l \rho_l\right)^2}$$
$$\longrightarrow M_{ij}^\epsilon(\rho) = \frac{\lambda_i \rho_i^+ \sum_l (\lambda_l - \lambda_j)\rho_l^+}{\left(\epsilon \vee \sum_l \lambda_l \rho_l^+\right)^2}.$$

Step 1/3: Existence to an approximate PDS when $\mu_0 \in L^2(\mathbb{R})$

- $\rho \mapsto M^\epsilon(\rho)$ locally Lipschitz and bounded $\rightarrow \exists!$ solution p_m^ϵ to a projection of the equation in dimension m.
- coercivity uniform in ϵ under (C) : $\exists$ solution p^ϵ satisfying uniform in ϵ EE by taking the limit $m \rightarrow \infty$.
- Taking p_ϵ^- as test function, we show that $p_\epsilon \geq 0$.
- $\forall \epsilon, \forall i, \sum_j M_{ji}^\epsilon = 1 \implies \sum_j p_j^\epsilon$ solves the heat equation $\longrightarrow$ lower bound uniform in ϵ (but not t, x) for $\sum_j \lambda_j p_j^\epsilon$.
- $\epsilon \rightarrow 0$, existence of a solution to the original PDS.

Step 2/3: Existence to the PDS when $\mu_0 \in \mathcal{P}(\mathbb{R})$

- By mollification of μ_0, we use the results of Step 1 to extract a solution to the PDS when $\mu_0 \in \mathcal{P}(\mathbb{R})$.
- We use the fact that $\sum_j p_j$ is solution to the heat equation to control the rate of explosion of $t \mapsto \int_{\mathbb{R}} \sum_{i=1}^{d} p_i^2(t,x)dx$ as $t \to 0$ uniformly in the mollification parameter.

Step 3/3: Existence of a weak the SDE (FBM)

Theorem (Figalli (2008))

*For $a : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ and $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ meas. and bounded let $L_t\varphi(x) = \frac{1}{2}a(t,x)\varphi''(x) + b(t,x)\varphi'(x)$.
If $[0,T] \ni t \mapsto \mu_t \in \mathcal{M}_+(\mathbb{R})$ is weakly continuous and solves the Fokker-Planck equation $\partial_t \mu_t = L_t^* \mu_t$ in the sense of distributions then there exists a probability measure P on $C([0,T],\mathbb{R})$ with marginals $(P_t = \mu_t)_{t\in[0,T]}$ such that
$\forall \varphi \in C_b^2(\mathbb{R}),\ \varphi(X_t) - \int_0^t L_s\varphi(X_s)ds$ is a P-martingale.*

$\Rightarrow$for $i \in \{1,\dots,d\}$, there exists a probab. P^i on $C([0,T],\mathbb{R})$ with $P_0^i = \mu_0$ and $P_t^i = \frac{p_i(t,x)dx}{\alpha_i}$ for $t \in (0,T]$ and $\forall \varphi \in C_b^2(\mathbb{R})$,

$$\varphi(X_t) - \int_0^t \frac{f^2(y_i)\sum_{j=1}^d p_j}{\sum_{j=1}^d f^2(y_j)p_j}(s,X_s)\varphi''(X_s)ds \text{ is a } P^i\text{-martingale.}$$

$P(dX,dY) = \sum_{i=1}^d \alpha_i P^i(dX) \otimes \delta_{y_i}(dY)$ weak solution.

Outline

Presentation

- We consider the following dynamics (RSLV):

$$dS_t = rS_t dt + \frac{f(Y_t)}{\sqrt{\mathbb{E}\left[f^2(Y_t)|S_t\right]}}\sigma_{Dup}(t, S_t)S_t dW_t,$$

  where $(Y_t)_{t\geq 0}$ takes values in $\mathcal{Y}$, and

$$\mathbb{P}\left(Y_{t+dt} = y_j | Y_t = y_i, S_t = x\right) = q_{ij}(x)dt.$$

- Switching diffusion, special case of LSV model.

- Jump distributions and intensities are functions of the asset level.

Assumptions

- (C), (Coerc. 1): f satisfies condition (C).
- (HQ), (Bounded I) $\exists \bar{q} > 0,\ s.t.\ \forall x \in \mathbb{R}, |q_{ij}(x)| \leq \bar{q}$.

We define $\tilde{\sigma}_{Dup}(t,x) := \sigma_{Dup}(t, e^x)$.

- (H1), (Bounded vol.) $\tilde{\sigma}_{Dup} \in L^\infty([0,T], W^{1,\infty}(\mathbb{R}))$.
- (H2), (Coerc. 2) $\exists \underline{\sigma} > 0$ s.t. $\underline{\sigma} \leq \tilde{\sigma}_{Dup}$ a.e. on $[0,T] \times \mathbb{R}$,.
- (H3), (Regul. 1) $\exists \eta \in (0,1], \exists H_0 > 0$, s.t. $\forall s,t \in [0,T], \forall x,y \in \mathbb{R}$,

$$|\tilde{\sigma}_{Dup}(s,x) - \tilde{\sigma}_{Dup}(t,y)| \leq H_0 \left(|x-y|^\eta + |t-s|^\eta\right).$$

(HQ), (H1) and (H2) permit to generalize the energy estimations to the Fokker-Planck system associated with $((\ln(S_t), Y_t))_{t\in[0,T]}$
With (H3), uniqueness and Aronson estimates for the Fokker-Planck equation associated with $(\ln(S_t^{Dup}))_{t\in[0,T]}$ where

$$dS_t^{Dup} = \sigma_{Dup}(t, S_t^{Dup}) S_t^{Dup} dW_t + rS_t^{Dup} dt,\ S_0^{Dup} = S_0.$$

$\longrightarrow$ replaces the heat equation

Main results

Theorem

Under Conditions (H1)-(H3), (HQ) and (C) there exists a weak solution to the SDE (RSLV). Moreover, it has the same marginals as the solution to the local volatility SDE

$$dS_t^{Dup} = \sigma_{Dup}(t, S_t^{Dup}) S_t^{Dup} dW_t + rS_t^{Dup} dt, \;\; S_0^{Dup} = S_0.$$

- We generalize the results of Figalli to the regime switching case.
- Uniqueness?

Main obstacle to deal with usual LSV models

Corresponding generalized (FBM) equation

$$dX_t = \frac{f(Y_t)}{\sqrt{\mathbb{E}\left[f^2(Y_t)|X_t\right]}}dW_t$$

$$dY_t = \eta(Y_t)(\rho dW_t + \sqrt{1-\rho^2}dB_t) + b(Y_t)dt$$

with $(B_t)_{t\geq 0}$ Brownian motion indep. of $(W_t)_{t\geq 0}$.
Typically $|\rho| \neq 1$ and (X_t, Y_t) should admit a density $p(t,x,y)$.
Even when $0 < \inf f \leq \sup f < \infty$, in the divergence form of the term

$$\partial_{xx}\left(\frac{f^2(y)\int p(t,x,z)dz}{\int f^2(z)p(t,x,z)dz}p(t,x,y)\right) = \partial_x\left(\frac{f^2(y)\int p(t,x,z)dz}{\int f^2(z)p(t,x,z)dz}\partial_x p(t,x,y)\right)$$
$$+\partial_x\left(\frac{f^2(y)p(t,x,y)}{\int f^2(z)p(t,x,z)dz}\int \partial_x p(t,x,z)dz - \frac{f^2(y)p(t,x,y)\int p(t,x,z)dz}{(\int f^2(z)p(t,x,z)dz)^2}\int f^2(z)\partial_x p(t,x,z)dz\right)$$

the red factors replacing $\frac{f^2(y_i)p_i(t,x)}{\sum_{j=1}^d f^2(y_j)p_j(t,x)}$ and $\frac{f^2(y_i)p_i(t,x)\sum_{j=1}^d p_j(t,x)}{(\sum_{j=1}^d f^2(y_j)p_j(t,x))^2}$ are no longer bounded (same problem for kernel approximations of $\mathbb{E}\left[f^2(Y_t)|X_t\right]$).

Thank you for your attention!