Pricing CVA adjustments: An expansion approach for WWR

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Introduction

- *XVA* is one of the most demanding issues in terms of prices and Greeks calculations
  - global portfolio pricing and collateral netting
  - incremental charging and reallocation
  - management of cross-asset and hybrid risks (eg. *WWR*).
- Consistency with spot prices is required (reusing validated and proven pricers)
- We show in Iben Taarit (2015) how to upgrade existant pricers in order to account for *WWR*

A brief reminder on *XVA* (cont')

- At valuation time 0, we define the Bilateral Credit Valuation Adjustment (BCVA) as seen by the the bank B as

$$BCVA(0) = LGD_C \mathbb{E}\left[1_{\{\tau_C \le \tau_B\}} 1_{\{\tau_C < T\}} D(0, \tau_C) \left(NPV(\tau_C, T) \right)^{+} \right] \tag{1}$$

- Analog formula for the Bilateral Debt Valuation Adjustment (DVA)
- A classical simplifying assumption consists in considering the **default of only one counterparty**

$$UCVA(0, T) = LGD_C \mathbb{E}\left[1_{\{\tau_C < T\}} D(0, \tau_C) \left(NPV(\tau_C, T) \right)^{+} \right] \tag{2}$$

- Unilateral adjustments = **mutual exclusion** of defaults

Pricing framework

- **Main goal**: approximated price (fast/accurate) of a contingent claim $h(X_T)$ subject to the default of the supplying party, i.e. $\tau < T$
- Stochastic intensity model for the default time τ

$$\begin{cases} \tau^{\epsilon} & = \inf\left\{t > 0 | \int_0^t \lambda_s^{\epsilon} ds > \xi\right\} \\ d\lambda_t^{\epsilon} & = \kappa(t)\left(\psi(t) - \lambda_t^{\epsilon}\right) dt + \epsilon \nu\left(t, \lambda_t^{\epsilon}\right) dW_t \\ \lambda_0^{\epsilon} & = \lambda_0 > 0 \end{cases} \tag{3}$$

where W :$\mathbb{R}-$valued SBM, $\xi \overset{\mathcal{L}}{\rightarrow} \mathcal{E}(1)$, $\epsilon \in [0, 1]$

- In addition, let $(X_t)_{t \geq 0}$ be a $\mathbb{R}^n-$valued diffusion process governed by

$$dX_t = (\Phi(t) + \Theta(t) X_t) dt + \Sigma(t) dB_t \ , X_0 \in \mathbb{R}^n \tag{4}$$

with $\Phi : [0, T] \rightarrow \mathbb{R}^n$, $\Theta : [0, T] \rightarrow \mathbb{R}^{n \times n}$, $\Sigma : [0, T] \rightarrow \mathbb{R}^{n \times d}$ and $(B_t)_{t \geq 0}$ a $\mathbb{R}^d-$valued SBM.

Pricing framework (cont')

- We define the instantaneous correlations $\rho = (\rho_i)_{i=1\ldots d}$ such that

$$d \langle W, B_i \rangle_t = \rho_i dt \ , \ 1 \leq i \leq d \tag{5}$$

- $\rho \neq 0$ and $\nu(t, \lambda_t^\epsilon) \neq 0 \Rightarrow$ Wrong-way risk
- Recovery in market value convention (recovery rate δ)

$$u_{h,\delta}^\epsilon(0, T) = \mathbb{E}\left[h(X_T)1_{\{\tau > T\}} + (1-\delta)\, u_{h,\delta}^\epsilon\left(\tau^-, T\right) 1_{\{\tau \leq T\}}\right] \tag{6}$$

- Duffie and Singleton (1999)

$$u_{h,\delta}^\epsilon(0, T) = \mathbb{E}\left[\exp\left(-(1-\delta)\int_0^T \lambda_t^\epsilon dt\right) h(X_T)\right] \tag{7}$$

Pricing methodology

Main Objective

$$u_{h,\delta}^{\epsilon=1}(S,T) = u_{h,\delta}^{\epsilon=0}(S,T) + \text{weighted sum of Greeks of } \mathbb{E}\left[h\left(X_T\right)\right] + Error$$

where

- $u_{h,\delta}^{0}(T) = e^{-(1-\delta)\int_0^T \lambda_t^0 dt}\mathbb{E}\left[h\left(X_T\right)\right]$ (classical pricing)
- the weighted sum of Greeks of $\mathbb{E}\left[h\left(X_T\right)\right]$ is given by the system

- we want the accuracy **to be bounded** using
  - The regularity of the intensity process λ_t^ϵ
  - The dependence of $u_{h,\delta}^\epsilon$ on $\int \lambda_s^\epsilon ds$

Comparison with similar works

- Expansion approach for credit intensity diffusion already addressed in Muroi 2005 and Muroi 2012
  - Linearization of the a payoff function $\Phi\left(r^{\epsilon}\left(T\right),\lambda^{\epsilon}\left(T\right)\right)$ (smoothness requirements)
- We follow Benhamou *et al.* (2009, 2010a,b). However, the setting is **fundamentally different**.
  - We perform expansion for $e^{-(1-\delta)\int_0^T \lambda_s^{\epsilon} ds}$
  - **Minimal dependence** on the regularity of h

Theorem (Second order approximation)

Under regularity assumption of the drift and diffusion of (λ_t^ϵ) *, one has*

$$
\begin{aligned}
u_{h,\delta}^{\epsilon=1}(0,T) = {} & u_{h,\delta}^{\epsilon=0}(0,T) + (1-\delta)^2\, C_{0,1}(T)\, u_{h,\delta}^{\epsilon=0}(0,T) \\
& - (1-\delta)\, C_{1,1}(T).Greek^{(1)}(T, X_T) \\
& - (1-\delta)\left(C_{2,1}(T) - (1-\delta)\, C_{2,2}(T)\right).Greek^{(2)}(T, X_T) \\
& + Error_2^{\epsilon=1}
\end{aligned}
$$

- $C_{0,1}(T) = \frac{1}{2}\int_0^T \left(\int_t^T e^{-\int_0^s \kappa(u)du} ds\, \nu(t)\right)^2 dt$
- $[C_{1,1}(T)]_i = \int_0^T \left(\int_t^T e^{-\int_0^s \kappa(u)du} ds\right) \nu(t)\, [\Sigma(t,T)\,\rho]_i\, dt$
- $[C_{2,1}(T)]_{i,j} =$
  $\int_0^T \left(\int_t^T \left(\int_s^T e^{-\int_0^u \kappa(v)dv} du\right) \nu^{(1)}(s)\, [\Sigma(s,T)\,\rho]_i\, ds\right) \nu(t)\, [\Sigma(t,T)\,\rho]_j\, dt$
- $[C_{2,2}(T)]_{i,j} = \ldots$

Numerical experiments

- Log-normal diffusion of the spot S_t. Default parameters are

T	$r/q/d$	Σ
10	0	30%

- 3 models of λ_t^ϵ

$\mathcal{N}$	$\mathcal{LN}$	$\mathcal{C}$
$\nu\left(t, \lambda_t^\epsilon\right) = \bar{\nu}_n$	$\nu\left(t, \lambda_t^\epsilon\right) = \bar{\nu}_{ln}\lambda_t^\epsilon$	$\nu\left(t, \lambda_t^\epsilon\right) = \bar{\nu}_c\sqrt{\lambda_t^\epsilon}$

- 2 risk regimes

Mid Risk	λ_0	κ	ψ	ρ	$\bar{\nu}_n$	$\bar{\nu}_{ln}$	$\bar{\nu}_c$
	0.01	1	0.02	30%	1%	50%	20%

High Risk	λ_0	κ	ψ	ρ	$\bar{\nu}_n$	$\bar{\nu}_{ln}$	$\bar{\nu}_c$
	0.03	1.6	0.08	90%	3%	100%	50%

- Benchmarking *versus* Monte Carlo (Paths $= 10^5$, 24 steps/ year)

Contingent Call option

	K/S	CI	Proxy	2nd Order Exp.	3nd Order Exp.
V	80%	0.12%	1.22%	0.00%	0.00%
	100%	0.15%	1.42%	0.00%	0.00%
	120%	0.17%	1.55%	0.00%	0.00%
LN	80%	0.12%	1.22%	0.00%	0.00%
	100%	0.15%	1.37%	0.00%	0.00%
	120%	0.17%	1.54%	0.00%	0.00%
CIR	80%	0.12%	3.28%	0.52%	0.07%
	100%	0.14%	3.43%	0.54%	0.05%
	120%	0.17%	4.11%	0.65%	0.06%

(a) Relative Error: Mid risk parameters

	K/S	CI	Proxy	2nd Order Exp.	3nd Order Exp.
V	80%	0.12%	7.65%	0.00%	0.00%
	100%	0.14%	8.57%	0.00%	0.00%
	120%	0.17%	9.32%	0.00%	0.00%
LN	80%	0.11%	23.28%	2.34%	1.17%
	100%	0.13%	27.84%	2.84%	1.57%
	120%	0.16%	31.29%	3.13%	1.69%
CIR	80%	0.11%	40.50%	7.19%	3.72%
	100%	0.12%	47.17%	8.55%	4.55%
	120%	0.15%	54.05%	9.76%	5.41%

(b) Relative Error: High risk parameters

A Wrong-way risk adjustment for CVA/DVA

- *UCVA* is usually approximated by

$$\begin{aligned} UCVA(0) &\approx \mathrm{LGD}_C \sum_{m=1}^{M} D(0,T_m)\,\mathbb{E}\left[\left(1_{T_{m-1}<\tau_C} - 1_{T_m<\tau_C}\right)\left(V(T_m, X_{T_m})\right)^+\right] \\ &\approx \mathrm{LGD}_C \sum_{m=1}^{M} D(0,T_m)\left(u^{\epsilon}_{V^+}(T_{m-1},T_m) - u^{\epsilon}_{V^+}(T_m,T_m)\right) \end{aligned}$$

where

$$u^{\epsilon}_{V^+,0} : (s,t) \mapsto \mathbb{E}\left[e^{-\int_0^s \lambda^{\epsilon}_{\omega} d\omega}\left(V(t,X_t)\right)^+\right]$$

Consequence

With $\delta = 0$ and $h = V^+$, our approximation formulas yield

$$UCVA(0) = UCVA^0(0) + \sum_{m=1}^{M} \text{weighted sum of Greeks of } \mathbb{E}\left[\left(V(X_{T_m})\right)^+\right] + \text{Error}$$

$$\begin{aligned} \text{Wrong-way Risk Adjustment} &= UCVA(0) - UCVA^0(0) \\ &= \text{Weighted sum of } \textit{exposure} \text{ Greeks} \end{aligned}$$

A WWR adjustment in the bilateral framework

We apply the same methodology

$$
\begin{aligned}
&BCVA(0)\\
&\approx \mathrm{LGD}_C \sum_{m=0}^{M} D(0,T_m)\,\mathbb{E}\left[\left(\mathbf{1}_{\{\tau_C \geq T_{m-1}\}} - \mathbf{1}_{\{\tau_C \geq T_m\}}\right)\mathbf{1}_{\{\tau_B > T_m\}}\left(V(T_m, X_{T_m})\right)^{+}\right]\\
&\approx \mathrm{LGD}_C \sum_{m=1}^{M} D(0,T_m)\left(u^{\epsilon}_{V^{+},0}(T_{m-1},T_m) - u^{\epsilon}_{V^{+},0}(T_m,T_m)\right)
\end{aligned}
$$

where

$$
\begin{aligned}
u^{\epsilon}_{V^{+},0}(s,t) &= \mathbb{E}\left[\mathbf{1}_{\{\tau_C \geq s\}}\mathbf{1}_{\{\tau_B \geq t\}}\left(V(t,X_t)\right)^{+}\right]\\
&= \mathbb{E}\left[\left(e^{-\int_0^s \lambda^{C,\epsilon}_{\omega} d\omega} e^{-\int_0^t \lambda^{B,\epsilon}_{\omega} d\omega}\right)\left(V(t,X_t)\right)^{+}\right]
\end{aligned}
$$

Consequence

With $\delta = 0$ and $h = V^{+}$, our approximation formulas yield

$$
BCVA(0) = UCVA^{0}(0) + \sum_{m=1}^{M} \text{weighted sum of Greeks of } \mathbb{E}\left[\left(V(X_{T_m})\right)^{+}\right] + \text{Error}
$$

Bilateral Wrong-way Risk $= UWWR_C + UWWR_B +$ First to default Risk

References I

Benhamou, Eric, Gobet, Emmanuel, & Miri, Mohamed. 2009. Smart expansion and fast calibration for jump diffusions. *Finance and stochastics, vol.13(4), pp.563-589*, **13**(4), 563–589.

Benhamou, Eric, Gobet, Emmanuel, & Miri, Mohamed. 2010a. Expansion formulas for European options in a local volatility model. *International journal of theoretical and applied finance, vol.13(4), pp.602-634*, **13**(04), 603–634.

Benhamou, Eric, Gobet, Emmanuel, & Miri, Mohamed. 2010b. Time Dependent Heston Model. *Siam journal on financial mathematics, vol.1, pp.289-325*, **1**(1), 289–325.

Iben Taarit, Marouan. 2015. Pricing derivatives with credit risk: Expansion formulas for stochastic intensity models. *Preprint*.

Muroi, Yoshifumi. 2005. Pricing contingent claims with credit risk: Asymptotic expansion approach. *Finance and stochastics*, **9**(3), 415–427.

Muroi, Yoshifumi. 2012. Pricing credit derivatives using an asymptotic expansion approach. *The journal of computational finance*, **15**(3), 135.