Asymmetric information in trading against disorderly liquidation of a large position.

Caroline Hillairet [1]   Cody Hyndman [2]   Ying Jiao [3]   Renjie Wang[2]

[1]ENSAE ParisTech Crest, France

[2]Concordia University, Canada

[3]Université Lyon 1, France

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Overview

Overview

Introduction

- We are interested in the following question:

  Is it possible to profit from the knowledge that a market participant, with large positions in a stock or derivative, will be forced to liquidate some or all of its position if the price crosses a certain threshold?

- Previous literature:
  - ◇ Insider trading, asymmetric information, and market manipulation trading strategies (see Kyle [11], Back [3] and Jarrow [8, 9].)
  - ◇ Liquidity models (see Gökay et al. [7])

- We are concerned with disorderly, rather than optimal, liquidation and from the view point of other market participants rather than that of the large trader or hedge fund.

Model setup

- A hedge fund holds a large position on a risky asset (such as stock) over an investment horizon $[0, T]$.

- The interest rate $r = 0$ and the risky asset price is modelled by the diffusion process
$$dS_t = S_t(\mu dt + \sigma dW_t), \quad 0 \leq t \leq T \tag{1.1}$$
where μ and σ are both constants.

- We denote by $\mathbb{F}$ the augmented filtration generated by W.

- Liquidation is triggered when the asset price passes below a certain level αS_0 for $\alpha \in (0, 1)$.

- The liquidation time τ is modelled as a first passage time of S
$$\tau := \inf\{t \geq 0, \ S_t \leq \alpha S_0\}.$$

Market impact

- We model market impact by a function $g(\Delta t; \Theta, K)$ of the form (similar to Li et al. [12])

$$g(\Delta t; \Theta, K) = 1 - \frac{K\Delta t}{\Theta}e^{1-\frac{\Delta t}{\Theta}}, \quad \Delta t \geq 0$$

where Δt stands for the amount of time after liquidation, i.e. $\Delta t = t - \tau$.

- We denote by $S_t^I(u)$ the risky asset price at time t after the liquidation time $\tau = u$ and

$$S_t^I(u) = g(t - u; \Theta, K)S_t, \quad u \leq t \leq T. \tag{1.2}$$

Impact function

- We may make the impact function more flexible with more parameters

$$g(\Delta t; \Theta_1, \Theta_2, K_1, K_2) = \begin{cases} 1 - \frac{(K_1 + K_2)\Delta t}{\Theta_1} e^{1 - \frac{\Delta t}{\Theta_1}} & 0 \leq \Delta t < \Theta_1, \\ 1 - K_1 - \frac{K_2(\Delta t + \Theta_2 - \Theta_1)}{\Theta_2} e^{1 - \frac{\Delta t + \Theta_2 - \Theta_1}{\Theta_2}} & \Theta_1 \leq \Delta t. \end{cases}$$
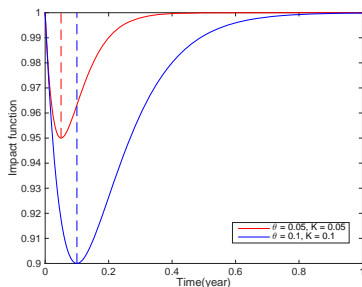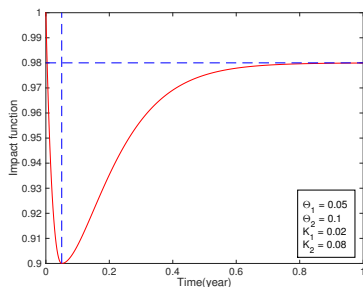

Figure: $g(\Delta t)$ with 2 parameters

Figure: $g(\Delta t)$ with 4 parameters

Parameters of impact function

- Θ determines the deviation and reversion speed.
- K controls the magnitude of the temporary market impact.
- There are multiple factors that influence the market impact magnitude and speed.
  - ◇ the size of the position to be liquidated
  - ◇ daily average volume
  - ◇ market depth and resiliency
  - ◇ the informational content of liquidation
  - ◇ other factors that might be known to, or estimated by, sufficiently informed investors
- We suppose Θ and K are random variables independent of $\mathbb{F}$ with support $(0, +\infty) \times (0, 1)$. The joint probability density is $\varphi(\theta, k)$.

Dynamics of asset price

- For any $u \geq 0$ we apply Itô's formula to (1.2) to find that

$$dS_t^I(u) = S_t^I(u) \left\{ \mu_t^I(u, \Theta, K)dt + \sigma dW_t \right\}, \quad t \geq u \qquad (1.3)$$

  where $\mu_t^I(\tau, \Theta, K) = \mu + \frac{g'(t-\tau; \Theta, K)}{g(t-\tau; \Theta, K)}$.

- Combining the asset price before and after liquidation, we may decompose the price process over the investment horizon $[0, T]$ as

$$S_t^M = 1_{\{t < \tau\}} S_t + 1_{\{t \geq \tau\}} S_t^I(\tau).$$

- Using (1.1) and (1.3) we obtain

$$dS_t^M = S_t^M \left\{ \mu_t^M(\Theta, K)dt + \sigma dW_t \right\} \qquad (1.4)$$

  where $\mu_t^M(\Theta, K) = 1_{\{t < \tau\}} \mu + 1_{\{t \geq \tau\}} \mu_t^I(\tau, \Theta, K)$.

- Example: with model parameters $S_0 = 80, \mu = 0.07, \sigma = 0.2, \alpha = 0.9, \Theta = 0.05, K = 0.1$. We illustrate the market impact on the drift term and the asset price in the figures below.
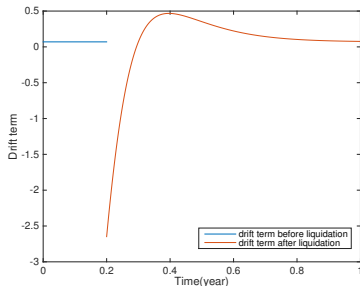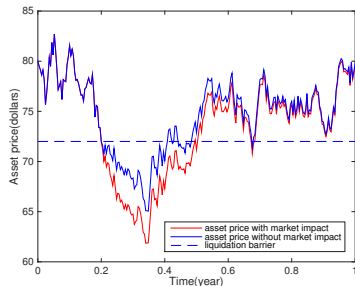

Figure: Drift term

Figure: Asset price

Overview

Three types of investors

- We classify market participants into three types according to different levels of information accessible to them.

  ⋄ **Fully informed investors:** have complete knowledge of the liquidation mechanism including the liquidation trigger level α, the functional form of the impact function as well as the realized value of (Θ, K).

  ⋄ **Partially informed investors:** know the liquidation trigger level α and the functional form of the impact function. They do not know the realized values of (Θ, K) but only the distribution of (Θ, K).

  ⋄ **Uninformed investors:** are not aware of the liquidation mechanism. They erroneously believe the asset price process always follows the dynamics of asset price without price impact.

Information accessible to three types of investors

- All three types of investors can observe the risky asset price S^M. Denote by $\mathbb{G}$ the augmented filtration generated by S^M, that is
$$\mathcal{G}_t = \sigma(S_v^M : 0 \leq v \leq t).$$

  ◇ Fully informed investors: $\mathbb{H} = (\mathcal{G}_t \vee \sigma(\Theta, K) : 0 \leq t \leq T)$
  ◇ Partially informed investors: $\mathbb{G}$ + knowledge on the distribution of (Θ, K).
  related literature : weak information (Baudoin [4]), Utility Maximization under partial Observations (Karatzas and Xue (1991), Karatzas and Zhao; Lefevre, Oksendal and Sulem (2000); Pham and Quenez (2001)...)

  ◇ Uninformed investors: $\mathbb{G}$.

- The liquidation time τ is $\mathbb{G}$-predictable stopping time (for any fixed α).
- Liquidation is observable to fully and partially informed investors since they know the value of α.
- Uninformed investors are not aware of the liquidation and know nothing about the liquidation trigger mechanism. They act as Merton-type investors.

Fully informed investors

- Fully informed investors' investment strategy is characterized by an $\mathbb{H}$-predictable process $\pi^{(2)}$ which represents the proportion of wealth invested in the risky asset.

- The corresponding wealth process $X^{(2)}$ satisfies the self financing dynamics

$$dX_t^{(2)} = X_t^{(2)} \pi_t^{(2)} \left(\mu_t^M(\Theta, K) dt + \sigma dW_t \right), \qquad 0 \le t \le T$$

- The admissible strategy set $\mathcal{A}^{(2)}$ is a collection of $\pi^{(2)}$ such that, for any $(\theta, k) \in (0, +\infty) \times (0, 1)$, almost surely

$$\int_0^T |\pi_t^{(2)} \mu_t^M(\theta, k)| dt + \int_0^T |\pi_t^{(2)} \sigma|^2 dt < \infty. \qquad (2.1)$$

Optimization problem (fully informed investors)

- Let $U(x)$ be a utility function satisfying the usual conditions.
- We formulate the optimization problem for fully informed investors:

$$\sup_{\pi^{(2)} \in \mathcal{A}^{(2)}} \mathbb{E}\left[U\left(X_T^{(2)} \right) \right]. \tag{2.2}$$

- By taking the initial information of (Θ, K) into consideration we may consider the following optimization problem

$$V_0^{(2)}(\Theta, K) := \sup_{\pi^{(2)} \in \mathcal{A}^{(2)}} \mathbb{E}\left[U\left(X_T^{(2)} \right) | \mathcal{H}_0 \right]. \tag{2.3}$$

  where $\mathcal{H}_0 = \sigma(\Theta, K)$.

- The link between the optimization problems (2.2) and (2.3) is given by Amendinger et al. [2]: an element of $\mathcal{A}^{(2)}$ attains the supremum in (2.2) if it attains the ω-wise optimum (2.3).

Optimal utility (fully informed investors)

- Martingale representation theorem for $(\mathbb{H}, \mathbb{P})$-local martingale (Amendinger [1])
- The optimization problem (2.3) can be solved by the "martingale approach" (see Karatzas and Shreve [10]).
- We define the martingale measure $\mathbb{Q}$ by the likelihood process

$$L_t := \frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{H}_t} = \exp\left\{ -\int_0^t \frac{\mu_v^M(\Theta, K)}{\sigma} dW_v - \int_0^t \frac{\left(\mu_v^M(\Theta, K)\right)^2}{2\sigma^2} dv \right\}.$$

- The optimal expected utility is given by

$$V_0^{(2)}(\Theta, K) = \mathbb{E}[U(I(\lambda L_T))|\mathcal{H}_0].$$

where $I = (U')^{-1}$ and λ is determined by

$$\mathbb{E}\left[I(\lambda L_T)L_T|\mathcal{H}_0\right] = X_0.$$

Power utility (fully informed investors)

- For power utility $U(x) = \frac{x^p}{p}$, $0 < p < 1$, the optimal expected utility is

$$V_0^{(2)}(\Theta, K) = \frac{(X_0)^p}{p} \left(\mathbb{E} \left[(L_T)^{\frac{p}{p-1}} \,\Big|\, \mathcal{H}_0 \right] \right)^{1-p}.$$

- Optimal strategy for the fully informed investors
  - On $[\tau \wedge T, T]$: Merton strategy

$$\hat{\pi}_t^{(2)} = \frac{\mu_t^I(\tau, \Theta, K)}{(1-p)\sigma^2}$$

  - On $[0, \tau \wedge T]$: Merton strategy+ "hedging demand for parameter risk"

$$\hat{\pi}_t^{(2)} = \frac{\mu}{(1-p)\sigma^2} + \frac{Z_t^H}{\sigma H_t}. \tag{2.4}$$

  with (H, Z^H) satisfying the BSDE

$$H_t = 1 + \int_t^T \left(\frac{p \left(\mu_v^M(\Theta, K) \right)^2}{2(1-p)^2 \sigma^2} H_v + \frac{p \mu_v^M(\Theta, K)}{(1-p)\sigma} Z_v^H \right) dv - \int_t^T Z_v^H dW_v. \tag{2.5}$$

Log utility (fully informed investors)

- For log utility $U(x) = \ln(x)$, the optimal expected utility is

$$V_0^{(2)}(\Theta, K) = \ln(X_0) - \mathbb{E}\left[\ln(L_T)|\mathcal{H}_0\right].$$

- The optimal strategy is simply the "myopic" Merton strategy:

$$\hat{\pi}_t^{(2)} = \frac{\mu_t^M(\Theta, K)}{\sigma^2}.$$

Explicit expression for optimal log utility

- The optimal log expected utility for fully informed investors is

$$
V_0^{(2)}(\Theta, K) =
$$
$$
\left\{ \mathcal{N}\left(\frac{-\frac{\ln \alpha}{\sigma} + (\frac{\mu}{\sigma} - \frac{1}{2}\sigma)T}{\sqrt{T}} \right) - \exp\left(\frac{2\mu}{\sigma^2} - \ln \alpha \right) \mathcal{N}\left(\frac{\frac{\ln \alpha}{\sigma} + (\frac{\mu}{\sigma} - \frac{1}{2}\sigma)T}{\sqrt{T}} \right) \right\} \times \left(\ln(X_0) + \frac{1}{2}(\mu - \frac{\mu^2}{\sigma^2})T \right)
$$
$$
+ \int_{\frac{\ln \alpha}{\sigma}}^{0} \int_y^{\infty} \frac{2\mu x(x - 2y)}{\sqrt{2\pi T^3}} \exp\left\{ (\frac{\mu}{\sigma} - \frac{1}{2}\sigma)x - \frac{1}{2}(\frac{\mu}{\sigma} - \frac{1}{2}\sigma)^2 T - \frac{1}{2T}(2y - x)^2 \right\} dxdy
$$
$$
- \frac{\ln \alpha}{\sigma} \int_0^T \frac{1}{\sqrt{2\pi t^3}} \exp\left\{ -\frac{1}{2t}\left(\frac{\ln \alpha}{\sigma} - (\frac{\mu}{\sigma} - \frac{1}{2}\sigma)t \right)^2 \right\} h^{(2)}(t; \Theta, K)dt
$$

where

$$
h^{(2)}(t; \Theta, K) := \ln X_0 + \frac{\mu \ln \alpha}{\sigma^2} + \frac{\mu}{2}t - \frac{\mu^2}{2\sigma^2}t + \int_t^T \left(\mu_v^I(t, \Theta, K) \right)^2 dv.
$$

Partially informed investors

- Recall that the asset price is given by

$$dS_t^M = S_t^M \left\{ \mu_t^M(\Theta, K)dt + \sigma dW_t \right\} \quad (2.6)$$

where $\mu_t^M(\Theta, K) = 1_{\{t < \tau\}}\mu + 1_{\{t \geq \tau\}}\mu_t^I(\tau, \Theta, K)$.

- The information accessible to partially informed investors is characterized by the filtration $\mathbb{G}$, however the drift term $\mu_t^M(\Theta, K)$ is not $\mathbb{G}$-adapted.

- Following Björk et al. [5] we define the innovation process $\tilde{W}$ by

$$d\tilde{W}_t = dW_t + \frac{\mu_t^M(\Theta, K) - \bar{\mu}_t^M}{\sigma}dt, \quad 0 \leq t \leq T$$

where

$$
\begin{aligned}
\bar{\mu}_t^M &= \mathbb{E}\left[\mu_t^M(\Theta, K) \big| \mathcal{G}_t \right] \\
&= 1_{\{t < \tau\}}\mu + 1_{\{t \geq \tau\}}\mathbb{E}\left[\mu_t^I(\tau, \Theta, K) | \mathcal{G}_t \right].
\end{aligned}
$$

Estimated drift term

- $\tilde{W}$ is a $(\mathbb{G}, \mathbb{P})$-Brownian motion.
- We may rewrite the asset price in (2.6) as

$$dS_t^M = S_t^M \left(\bar{\mu}_t^M dt + \sigma d\tilde{W}_t \right), \qquad 0 \le t \le T. \tag{2.7}$$

- The drift term $\bar{\mu}_t^M$ is $\mathbb{G}$-adapted. To find $\bar{\mu}_t^M$ we need to compute $\bar{\mu}_t^I := \mathbb{E}\left[\mu_t^I(\Theta, K) | \mathcal{G}_t \right]$, which is essentially a Bayesian problem.

$$\bar{\mu}_t^I = \frac{\int_0^\infty \int_0^1 \left\{ \mu_t^M(\theta, k) \exp \left\{ \int_0^t \frac{\mu_v^M(\theta,k)}{\sigma} dW_v + \int_0^t \frac{\left(\mu_v^M(\theta,k) \right)^2}{2\sigma^2} dv \right\} \right\} \varphi(\theta, k) d\theta dk}{\int_0^\infty \int_0^1 \left\{ \exp \left\{ \int_0^t \frac{\mu_v^M(\theta,k)}{\sigma} dW_v + \int_0^t \frac{\left(\mu_v^M(\theta,k) \right)^2}{2\sigma^2} dv \right\} \right\} \varphi(\theta, k) d\theta dk}. \tag{2.8}$$

Optimization problem (partially informed investors)

- The admissible strategy for partially informed investors is characterized by an $\mathbb{G}$-predictable process $\pi^{(1)}$ satisfying the integrability condition. The admissible strategy set is denoted by $\mathcal{A}^{(1)}$.

- The wealth process $X^{(1)}$ satisfies the dynamics

$$dX_t^{(1)} = X_t^{(1)} \pi_t^{(1)} \left(\bar{\mu}_t^M dt + \sigma d\tilde{W}_t \right), \quad 0 \leq t \leq T.$$

- The optimization problem is

$$V_0^{(1)} := \sup_{\pi^{(1)} \in \mathcal{A}^{(1)}} \mathbb{E}\left[U\left(X_T^{(1)} \right) \right]. \tag{2.9}$$

Optimal utility(partially informed investors)

- Martingale representation Theorem (Fujisaki et al. (1972)):
  any $(\mathbb{P}, \mathbb{G})$-local martingale can be represented as a stochastic integral with respect to $\tilde{W}$.

- We define the martingale measure $\bar{\mathbb{Q}}$ by the density process

$$\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}}\bigg|_{\mathcal{G}_t} := \bar{L}_t = \exp\left\{ -\int_0^t \frac{\bar{\mu}_v^M}{\sigma} d\tilde{W}_v - \int_0^t \frac{\left(\bar{\mu}_v^M\right)^2}{2\sigma^2} dv \right\}.$$

- The optimal expected utility is given by

$$V_0^{(1)} = \mathbb{E}[U(I(\lambda \bar{L}_T))].$$

where $I = (U')^{-1}$ and λ is determined by

$$\mathbb{E}\left[I(\lambda \bar{L}_T)\bar{L}_T\right] = x_0.$$

Power utility (partially informed investors)

- For power utility $U(x) = \frac{x^p}{p}$, $0 < p < 1$, the optimal expected utility is

$$V_0^{(1)} = \frac{(x_0)^p}{p} \left(\mathbb{E}\left[(\bar{L}_T)^{\frac{p}{p-1}} \right] \right)^{1-p}.$$

- The optimal strategy has the following explicit expression

$$\hat{\pi}_t^{(1,b)} = \frac{\mu}{(1-p)\sigma^2} + \frac{Z_t^{\bar{H}}}{\sigma \bar{H}_t}, \quad 0 \le t < \tau \wedge T, \qquad (2.10)$$

$$\hat{\pi}_t^{(1,a)} = \frac{\bar{\mu}_t^I}{(1-p)\sigma^2}, \quad \tau \wedge T \le t \le T. \qquad (2.11)$$

where $(\bar{H}, Z^{\bar{H}})$ satisfies the linear BSDE

$$\bar{H}_t = 1 + \int_t^T \left(\frac{p\left(\bar{\mu}_v^M\right)^2}{2(1-p)^2\sigma^2} \bar{H}_v + \frac{p\bar{\mu}_v^M}{(1-p)\sigma} Z_v^{\bar{H}} \right) dv - \int_t^T Z_v^{\bar{H}} d\tilde{W}_v. \qquad (2.12)$$

Log utility (partially informed investors)

- For log utility $U(x) = \ln x$, the optimal expected utility is

$$V_0^{(1)} = \ln(x_0) - \mathbb{E}\left[\ln(\bar{L}_T)\right].$$

- The optimal strategy is simply the "myopic" Merton strategy:

$$\hat{\pi}_t^{(1)} = \frac{\bar{\mu}_t^M}{\sigma^2}.$$

- Similar to the case of fully informed investors, we have explicit expression for the optimal expected utility $V_0^{(1)}$.

Uninformed investors

- Uninformed investors erroneously believe the risky asset price follows the a Black-choles dynamic with constant μ. They act as Merton investors.
- Merton strategies:
    - ⋄ power utility: $\pi_t^{(0)} = \frac{\mu}{(1-p)\sigma^2}$.
    - ⋄ log utility: $\pi_t^{(0)} = \frac{\mu}{\sigma^2}$.
- Regardless of uninformed investors' beliefs, the actual wealth process evolves according to the actual asset price and follows the dynamics

$$dX_t^{(0)} = X_t^{(0)} \pi_t^{(0)} \left\{ \mu_t^M(\Theta, K)dt + \sigma dW_t \right\}, \quad 0 < t \leq T. \quad (2.13)$$

- We also have explicit expression for $\mathbb{E}[U(X_T^{(0)})]$.

Overview

- We present some numerical results of the optimization problem for different types of investors using Monte Carlo simulation.

  ◇ We set $\mu = 0.07, \sigma = 0.2, S_0 = 80$ and $T = 1$.

  ◇ The liquidation trigger level is chosen as $\alpha = 0.9$.

  ◇ The stochastic processes are discretized using an Euler scheme with $N = 250$ steps and time intervals of length $\Delta t = \frac{1}{250}$.

  ◇ We suppose the random variables (Θ, K) have joint uniform distribution on $[0.05, 0.15] \times [0.02, 0.08]$.

  ◇ The number of simulations is 10^5.

For a realized $\Theta = 0.1, K = 0.05$, we compare the true drift term $\mu(\tau, \Theta, K)$ and the filtered estimate $\bar{\mu}$.
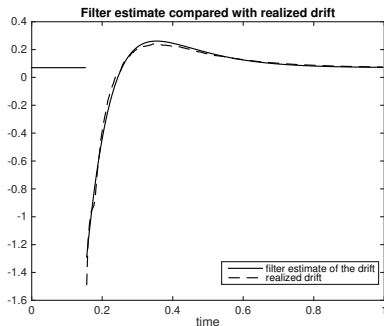

Figure: Filter estimate of the drift compared with the realized drift

Optimal utilities (power utility)

- For the specific power utility function $U(x) = 2x^{\frac{1}{2}}$, we calculate the optimal expected utilities for three types of investors.

Expected utilities	Numerical evaluation		
	Sample mean	Relative standard error	95% estimated confidence interval
Fully informed	48.9602	0.0883	[44.5223, 53.0279]
Partially informed	31.3099	0.0172	[30.7767, 31.8342]
Uninformed	18.9228	0.0012	[18.8796, 18.9661]

Table: Numerical evaluation of optimal power utilities for three types of investors

Optimal utilities (log utility)

- The optimal expected log utilities are also calculated.

Expected utilities	Numerical evaluation		
	Sample Mean	Relative standard error	95% estimated confidence interval
Fully informed	4.8282	0.0073	[4.8219, 4.8346]
Partially informed	4.7579	0.0080	[4.7520, 4.7638]
Uninformed	4.3665	0.0005	[4.3621, 4.3709]

Table: Numerical evaluation of optimal log utilities for three types of investors

Optimal strategies

- For power utility, the optimal strategies for fully and partially informed investors relies on the BSDE (2.5) and (2.12):
  - ⋄ recursive scheme using Monte Carlo regression (refer to Gobet et al. [6])

Asset market price over [0,T]

Optimal strategy for fully and partially informed investors over [0,T]
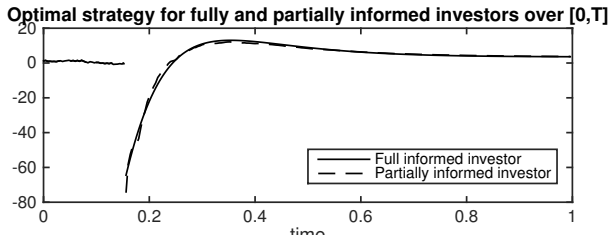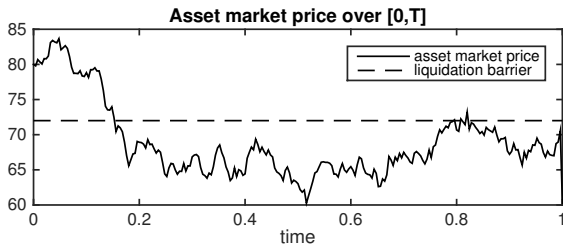
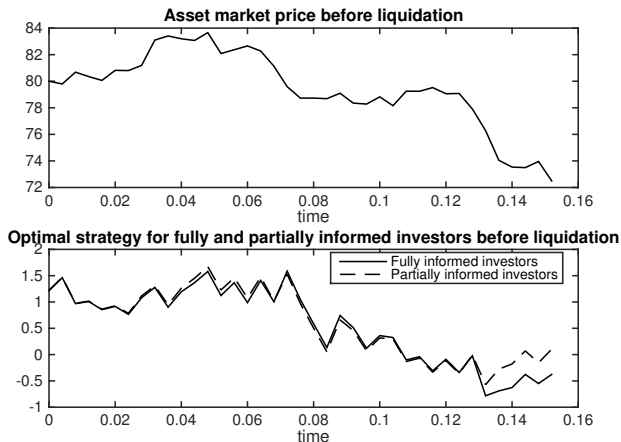# Optimal strategies before liquidation

Figure: Approximated optimal strategy for fully and partially informed investors before liquidation
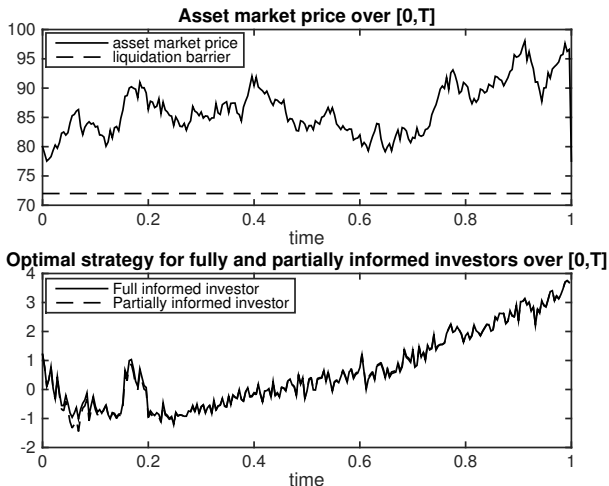
Optimal strategies without liquidation

Figure: Approximated optimal strategy for fully and partially informed investors without liquidation

The value of information about liquidation

- We can use the differences between full, partial, and no information to measure the value of access to information about liquidation impact.

- The model can be improved in many ways (ongoing).

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Thank you !

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