Bounds for VIX Futures given S&P 500 Smiles

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BOUNDS FOR VIX FUTURES GIVEN S&P 500 SMILES

JULIEN GUYON, ROMAIN MENEGAUX, AND MARCEL NUTZ

ABSTRACT. We derive sharp bounds for the prices of VIX futures using the full information of S&P 500 smiles. To that end, we formulate the model-free sub/superreplication of the VIX by trading in the S&P 500 and its vanilla options as well as the forward-starting log-contracts. A dual problem of minimizing/maximizing certain risk-neutral expectations is introduced and shown to yield the same value.

The classical bounds for VIX futures given the smiles only use a calendar spread of log-contracts on the S&P 500. We analyze for which smiles the classical bounds are sharp and how they can be improved when they are not. In particular, we introduce a family of functionally generated portfolios which often improves the classical bounds while still being tractable; more precisely, determined by a single concave/convex function on the line. Numerical experiments on market data and SABR smiles show that the classical lower bound can be improved dramatically, whereas the upper bound is often close to optimal.

1. INTRODUCTION

In this article, we derive sharp bounds for the prices of VIX futures by using the full information of S&P 500 smiles at two maturities. The VIX (short for volatility index) is published by the Chicago Board Options Exchange (CBOE) and used as an indicator of short-term options-implied volatility. By definition, the VIX is the implied volatility of the 30-day variance swap on the S&P 500; see [9]. Equivalently, using the well-known link between realized variance and log-contracts [19], the VIX at date T_1 is the implied volatility of a log-contract that delivers $\ln(S_{T_2}/S_{T_1})$ at $T_2 = T_1 + \tau$, where $\tau = 30$ days and S_{T_i} is the S&P 500 at date T_i:

$$(\mathrm{VIX}_{T_1})^2 = -\frac{2}{\tau}\mathrm{Price}_{T_1}\left[\ln\left(\frac{S_{T_2}}{S_{T_1}}\right)\right];$$

we are assuming zero interest rates, repos, and dividends for simplicity. The log-contract can itself be replicated at T_1 using call and put options on the S&P 500 with maturity T_2. The VIX index cannot be

Motivation

Objectives:

- Derive sharp bounds for the prices of VIX futures using the full information of S&P 500 smiles
- Derive the corresponding sub/superreplicating portfolios
- Test our results on market data: see if we improve the classical bounds and portfolios
- Characterize the market smiles for which the classical bounds are sharp
- Study specific families of smiles μ_1, μ_2 and corresponding portfolios

Reminder on the VIX

- VIX = Volatility IndeX
- Published every 15 seconds by the Chicago Board Options Exchange
- Indicator of short-term options-implied volatility
- Definition:
  VIX = the implied volatility of the 30-day variance swap on the S&P 500
- Equivalently, using the link between realized variance and log-contracts (Neuberger, Dupire, 1990-94):
  VIX at date T_1 = the implied volatility of a log-contract that delivers $\ln(S_2/S_1)$ at $T_2 = T_1 + \tau$ ($\tau = 30$ days):

$$V^2 \equiv (\mathrm{VIX}_{T_1})^2 \equiv -\frac{2}{\tau}\mathrm{Price}_{T_1}\left[\ln\left(\frac{S_2}{S_1}\right)\right]$$

- S_1 = S&P 500 at T_1, S_2 = S&P 500 at T_2, V = VIX at T_1
- We have assumed zero interest rates, repos, and dividends for simplicity
- The log-contract can itself be replicated at T_1 using OTM call and put options on the S&P 500 with maturity T_2

Reminder on VIX futures

- The VIX index cannot be traded, but VIX futures can
- VIX future expiring at T_1 = the instrument that pays $V \equiv \mathrm{VIX}_{T_1}$ at T_1
- $V^2 \equiv \mathrm{VIX}_{T_1}^2$ can be replicated using vanilla options on the S&P 500:

$$V^2 \equiv (\mathrm{VIX}_{T_1})^2 \equiv -\frac{2}{\tau}\mathrm{Price}_{T_1}\left[\ln\left(\frac{S_2}{S_1}\right)\right]$$

$\Longrightarrow$ To replicate exactly V^2 at time 0: buy $-\frac{2}{\tau}\ln S_2$, sell $-\frac{2}{\tau}\ln S_1$

- But its square root $V \equiv \mathrm{VIX}_{T_1}$ cannot
- Instead, sub/superreplication in the S&P 500 and its options leads to model-free lower/upper bounds on the price of the VIX future

Options on realized variance vs Options on implied variance

Options on realized variance	Options on implied variance
Call/put on realized variance	VIX future, call/put on VIX
Path-dependent option on underlying asset	Vanilla option on listed option prices on underlying asset
Skorokhod embedding problem	Martingale optimal transport
Carr and Lee, 2003 Dupire, 2005 Carr and Lee, 2008 Cox and Wang, 2013 ...	De Marco and Henry-Labordère, 2015

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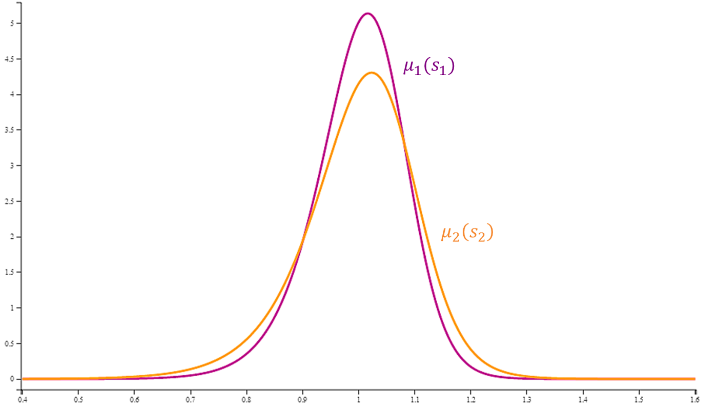
Carr and Lee, 2003 Dupire, 2005 Carr and Lee, 2008 Cox and Wang, 2013 ...	De Marco and Henry-Labordère, 2015 **This talk**

Notations and assumptions

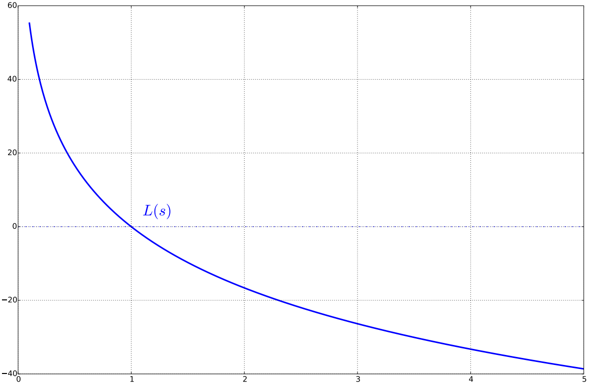
- $\mu_1 =$ risk-neutral distribution of $S_1 \longleftrightarrow$ market smile of S&P at T_1
- $\mu_2 =$ risk-neutral distribution of $S_2 \longleftrightarrow$ market smile of S&P at T_2



Notations and assumptions

- $L(x) = -\frac{2}{\tau}\ln x$: convex, decreasing

$$V^2 \equiv (\text{VIX}_{T_1})^2 \equiv -\frac{2}{\tau}\text{Price}_{T_1}\left[\ln\left(\frac{S_2}{S_1}\right)\right] = \text{Price}_{T_1}\left[L\left(\frac{S_2}{S_1}\right)\right]$$



Notations and assumptions

- We denote $\mathbb{E}^i \equiv \mathbb{E}^{\mu_i}$ and assume

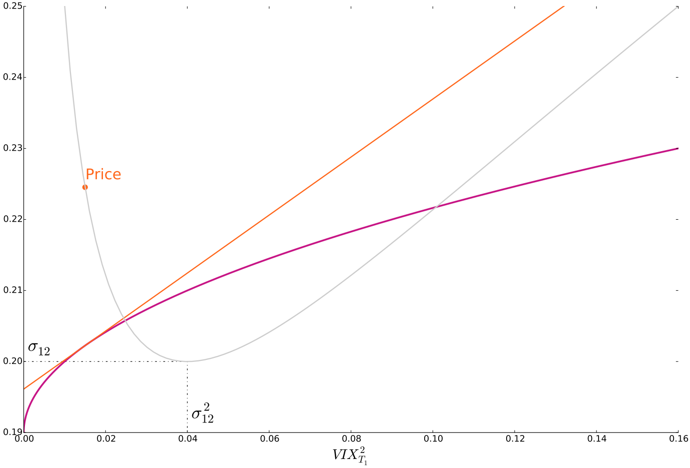
$$\mathbb{E}^i[S_i] = S_0, \qquad \mathbb{E}^i[|L(S_i)|] < \infty, \qquad i \in \{1,2\}$$

- We define

$$\begin{aligned} \sigma_{12}^2 &\equiv \mathbb{E}^2[L(S_2)] - \mathbb{E}^1[L(S_1)] \\ &= \mathrm{Price}_{T_0=0}\left[L\left(\frac{S_2}{S_1}\right)\right] \\ &= \mathrm{Price}_{T_0=0}[V^2] \end{aligned}$$

- No calendar arbitrage $\Longrightarrow \mu_1 \preceq \mu_2$ (convex order) $\Longrightarrow \sigma_{12}^2 \geq 0$
- σ_{12} is the implied volatility at time 0 of the forward-starting log-contract/variance swap on the S&P 500 starting at T_1 and maturing at T_2

Classical superreplication of VIX futures



Classical sub/superreplication of VIX futures

- Replicate exactly V^2 at time 0: buy $L(S_2)$, sell $L(S_1)$
- Classical upper bound $= \sigma_{12}$
- Classical lower bound $= 0$
- Concavity of the square root $\Longrightarrow$ Classical upper bound is good, classical lower bound is bad

1. LP formulation

Optimal sub/superreplication formulation: LP problem (De Marco, Henry-Labordère, 2015)

Available instruments:

- At time 0:
  - $u_1(S_1)$: vanilla payoff maturity T_1 (including cash)
  - $u_2(S_2)$: vanilla payoff maturity T_2
  - Cost: $\mathbb{E}^1[u_1(S_1)] + \mathbb{E}^2[u_2(S_2)]$
- At time T_1:
  - $\Delta^S(S_1, V)(S_2 - S_1)$: delta hedge
  - $\Delta^L(S_1, V)(L(S_2/S_1) - V^2)$: enter in $\Delta^L(S_1, V)$ log-contracts
  - Cost: 0

Superreplicating portfolio $\mathcal{U}_{\text{super}}$: for all $(s_1, s_2, v) \in (\mathbb{R}_+^*)^2 \times \mathbb{R}_+$

$$u_1(s_1) + u_2(s_2) + \Delta^S(s_1, v)(s_2 - s_1) + \Delta^L(s_1, v)\left(L\left(\frac{s_2}{s_1}\right) - v^2\right) \geq v$$

Optimal model-free no-arbitrage **upper bound**:

$$P_{\text{super}} \equiv \inf_{\mathcal{U}_{\text{super}}} \left\{\mathbb{E}^1[u_1(S_1)] + \mathbb{E}^2[u_2(S_2)]\right\}$$

LP problem with infinity of params $u_1, u_2, \Delta^S, \Delta^L$ and infinity of constraints

Optimal sub/superreplication formulation: LP problem (De Marco, Henry-Labordère, 2015)

Available instruments:

- At time 0:
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  - $\Delta^L(S_1, V)(L(S_2/S_1) - V^2)$: enter in $\Delta^L(S_1, V)$ log-contracts
  - Cost: 0

Subreplicating portfolio $\mathcal{U}_{\text{sub}}$: for all $(s_1, s_2, v) \in (\mathbb{R}_+^*)^2 \times \mathbb{R}_+$

$$u_1(s_1) + u_2(s_2) + \Delta^S(s_1, v)(s_2 - s_1) + \Delta^L(s_1, v)\left(L\left(\frac{s_2}{s_1}\right) - v^2\right) \leq v$$

Optimal model-free no-arbitrage **lower bound**:

$$P_{\text{sub}} \equiv \sup_{\mathcal{U}_{\text{sub}}} \left\{\mathbb{E}^1[u_1(S_1)] + \mathbb{E}^2[u_2(S_2)]\right\}$$

LP problem with infinity of params $u_1, u_2, \Delta^S, \Delta^L$ and infinity of constraints

Optimal sub/superreplication formulation: LP problem (De Marco, Henry-Labordère, 2015)

Available instruments:

- At time 0:
  - $u_1(S_1)$: vanilla payoff maturity T_1 (including cash)
  - $u_2(S_2)$: vanilla payoff maturity T_2
  - Cost: $\mathbb{E}^1[u_1(S_1)] + \mathbb{E}^2[u_2(S_2)]$
- At time T_1:
  - $\Delta^S(S_1, V)(S_2 - S_1)$: delta hedge
  - $\Delta^L(S_1, V)(L(S_2/S_1) - V^2)$: enter in $\Delta^L(S_1, V)$ log-contracts
  - Cost: 0

Subreplicating portfolio $\mathcal{U}_{\text{sub}}$: for all $(s_1, s_2, v) \in (\mathbb{R}_+^*)^2 \times \mathbb{R}_+$

$$u_1(s_1) + u_2(s_2) + \Delta^S(s_1, v)(s_2 - s_1) + \Delta^L(s_1, v)\left(L\left(\frac{s_2}{s_1}\right) - v^2\right) \leq v$$

Optimal model-free no-arbitrage lower bound:

$$P_{\text{sub}} \equiv \sup_{\mathcal{U}_{\text{sub}}} \left\{\mathbb{E}^1[u_1(S_1)] + \mathbb{E}^2[u_2(S_2)]\right\}$$

LP problem with infinity of params $u_1, u_2, \Delta^S, \Delta^L$ and infinity of constraints

Classical sub/superreplication of VIX futures

$$\Delta^S \equiv 0, \quad \Delta^L(s_1, v) = -a, \quad u_1(s_1) = -aL(s_1) + b, \quad u_2(s_2) = aL(s_2)$$

$$u_1(s_1) + u_2(s_2) + \Delta^S(s_1, v)(s_2 - s_1) + \Delta^L(s_1, v)\left(L\left(\frac{s_2}{s_1}\right) - v^2\right)$$
$$= -aL(s_1) + b + aL(s_2) - a\left(L\left(\frac{s_2}{s_1}\right) - v^2\right) = av^2 + b$$

does not depend on s_1, s_2: perfect replication of v^2

Classical (nonoptimal) superreplication: Minimize $\mathbb{E}^1[u_1(S_1)] + \mathbb{E}^2[u_2(S_2)]$ over all a, b such that $av^2 + b \geq v \Longrightarrow$

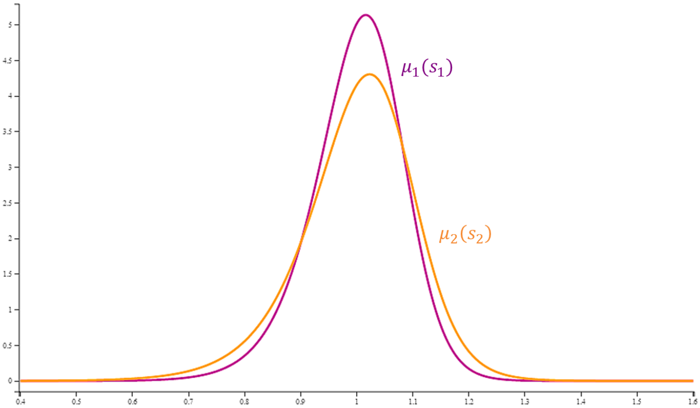
$$a^* = \frac{1}{2\sigma_{12}}, \qquad b^* = \frac{\sigma_{12}}{2}, \qquad \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^2[u_2(S_2)] = \sigma_{12}$$

Classical (nonoptimal) subreplication: Maximize $\mathbb{E}^1[u_1(S_1)] + \mathbb{E}^2[u_2(S_2)]$ over all a, b such that $av^2 + b \leq v \Longrightarrow$

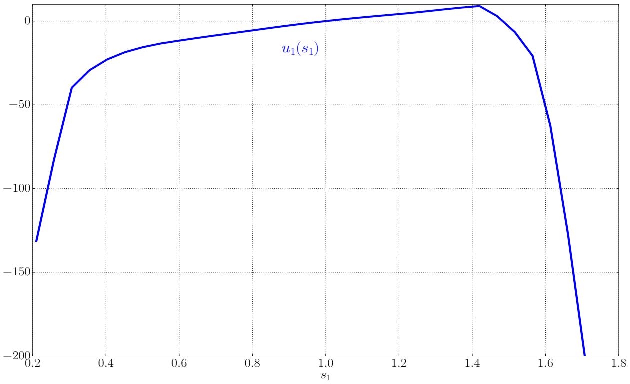
$$a^* = 0, \qquad b^* = 0, \qquad \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^2[u_2(S_2)] = 0$$

LP solver: numerical results

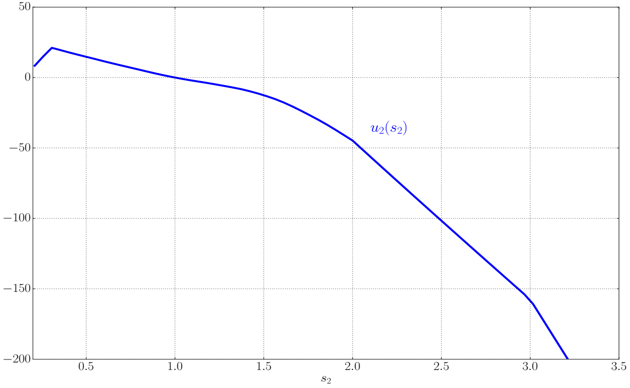
SABR risk-neutral densities



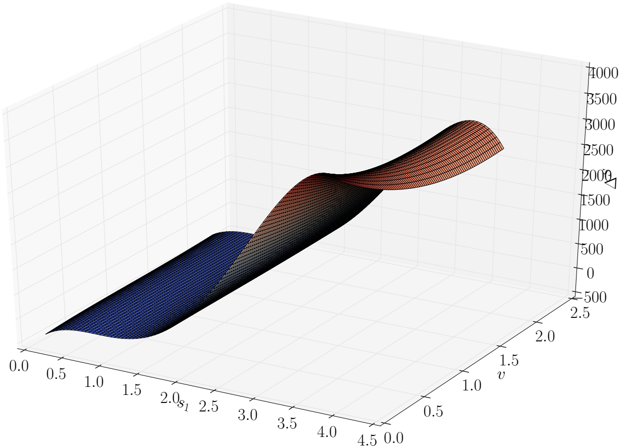
LP solver: numerical results for subreplication



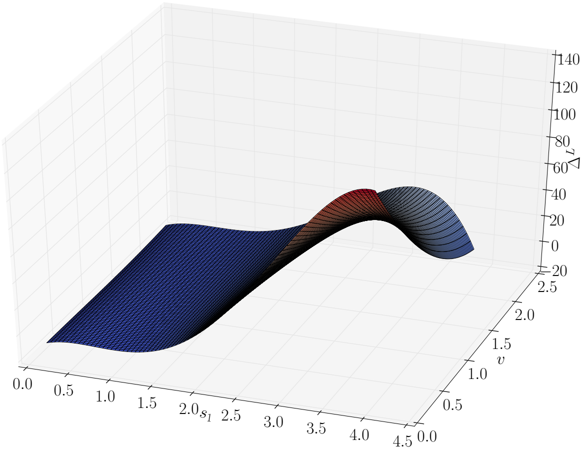
LP solver: numerical results for subreplication



LP solver: numerical results for subreplication

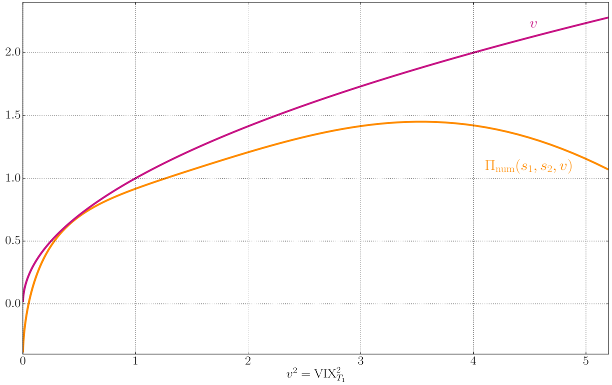


LP solver: numerical results for subreplication

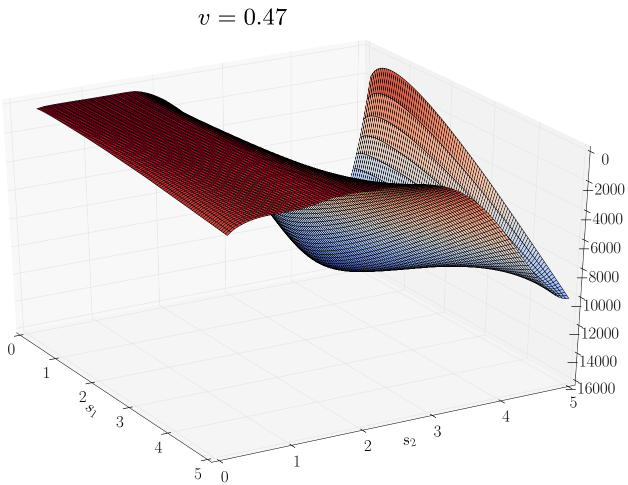


LP solver: numerical results for subreplication

$$s_1 = 1.03, \qquad s_2 = 1.18$$

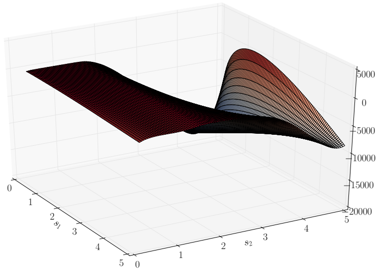


LP solver: numerical results for subreplication



LP solver: numerical results for subreplication

- Lower bound $= 7.2\% \gg 0 =$ classical lower bound
- Upper bound $= 22.8\% = \sigma_{12} =$ classical upper bound
- Practical problem: How do we try all $u_1(s_1)$, $u_2(s_2)$, $\Delta^S(s_1, v)$, $\Delta^L(s_1, v)$? How do we check the sub/superreplicating constraints **everywhere**?
- $\Longrightarrow$ Build a richer family of functionally generated sub/superreplicating portfolios, guaranteed to satisfy the constraints everywhere



2. A new family of functionally generated sub/superreplicating portfolios

A new family of functionally generated sub/superreplicating portfolios

For a **convex** function $\varphi : \mathbb{R}_+^* \times \mathbb{R} \to \mathbb{R}$ and $s_1 > 0$, we denote by

$$\varphi^*_{\text{super}}(s_1) \equiv \sup_{v \geq 0} \left\{ v - \varphi(s_1, L(s_1) + v^2) \right\}$$

the **smallest** function $u_1 : \mathbb{R}_+^* \to \mathbb{R} \cup \{+\infty\}$ such that

$$\forall s_1 > 0, \; \forall v \geq 0, \qquad u_1(s_1) + \varphi(s_1, L(s_1) + v^2) \geq v$$

Proposition

Let $\varphi : \mathbb{R}_+^* \times \mathbb{R} \to \mathbb{R}$ be a **convex** *function. The following portfolio* **superreplicates** *the VIX:*

$$\begin{aligned} u_1(s_1) &= \varphi^*_{\text{super}}(s_1), \qquad u_2(s_2) = \varphi(s_2, L(s_2)) \\ \Delta^S(s_1, v) &= -\partial_{1,r}\varphi(s_1, L(s_1) + v^2), \quad \Delta^L(s_1, v) = -\partial_{2,r}\varphi(s_1, L(s_1) + v^2) \end{aligned}$$

A new family of functionally generated sub/superreplicating portfolios

Proof.

Since φ is convex, φ is above tangent hyperplane:

$$\varphi(s_2, L(s_2)) - \varphi(s_1, L(s_1) + v^2) \geq$$
$$\partial_{1,r}\varphi(s_1, L(s_1) + v^2)(s_2 - s_1) + \partial_{2,r}\varphi(s_1, L(s_1) + v^2)(L(s_2) - L(s_1) - v^2)$$

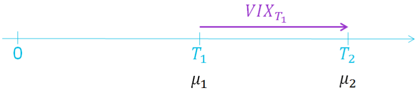
This yields

$$\begin{aligned} & u_1(s_1) + u_2(s_2) + \Delta^S(s_1, v)(s_2 - s_1) + \Delta^L(s_1, v)\left(L\left(\frac{s_2}{s_1}\right) - v^2\right) \\ = \; & u_1(s_1) + \varphi(s_2, L(s_2)) - \partial_{1,r}\varphi(s_1, L(s_1) + v^2)(s_2 - s_1) \\ & - \partial_{2,r}\varphi(s_1, L(s_1) + v^2)(L(s_2) - L(s_1) - v^2) \\ \geq \; & u_1(s_1) + \varphi(s_1, L(s_1) + v^2) \geq v \end{aligned}$$

□

A new family of functionally generated sub/superreplicating portfolios

$$\varphi(s_2, L(s_2)) - \varphi(s_1, L(s_1) + v^2) \geq$$
$$\partial_{1,r}\varphi(s_1, L(s_1) + v^2)(s_2 - s_1) + \partial_{2,r}\varphi(s_1, L(s_1) + v^2)(L(s_2) - L(s_1) - v^2)$$



Interpretation at T_1:

- Price of S_2 at T_1 is S_1
- Price of $L(S_2)$ at T_1 is $L(S_1) + V^2$
- $\Longrightarrow$ R.h.s. is costless
- $\Longrightarrow$ Price of $u_2(S_2) \equiv \varphi(S_2, L(S_2))$ at T_1 is $\geq \varphi(S_1, L(S_1) + V^2)$ ($\longleftrightarrow$ Jensen's inequality)
- We can exactly superreplicate $u_1(S_1) + \varphi(S_1, L(S_1) + V^2)$ at T_1
- Choose u_1 and φ such that this is always $\geq V$

A new family of functionally generated sub/superreplicating portfolios

Classical superreplication portfolio:

- $\varphi(x, y) = ay$
- $u_2(s_2) = aL(s_2)$, $\Delta^S(s_1, v) = 0$, and $\Delta^L(s_1, v) = -a$
- For $\varphi^*_{\text{super}}(s_1)$ to be finite, one must choose $a > 0$. Then

$$\varphi^*_{\text{super}}(s_1) = \frac{1}{4a} - aL(s_1) \equiv u_1(s_1)$$

- $\mathbb{E}^1[u_1(S_1)] + \mathbb{E}^2[u_2(S_2)] = \frac{1}{4a} - a\mathbb{E}^1[L(S_1)] + a\mathbb{E}^2[L(S_2)] = \frac{1}{4a} + a\sigma_{12}^2$
- Minimizing over the parameter a gives $a = \frac{1}{2\sigma_{12}}$ and we recover the classical superreplication portfolio
- **The proposition tells us that, in order to build analytical superreplicating portfolios, instead of considering a payoff $u_2(s_2)$ which is linear in $L(s_2)$, one can more generally consider convex functions of $(s_2, L(s_2))$**

A new family of functionally generated sub/superreplicating portfolios

For a **concave** function $\varphi : \mathbb{R}_+^* \times \mathbb{R} \to \mathbb{R}$ and $s_1 > 0$, we denote by

$$\varphi_{\text{sub}}^*(s_1) \equiv \inf_{v \geq 0} \left\{ v - \varphi(s_1, L(s_1) + v^2) \right\}$$

the **largest** function $u_1 : \mathbb{R}_+^* \to \mathbb{R} \cup \{-\infty\}$ such that

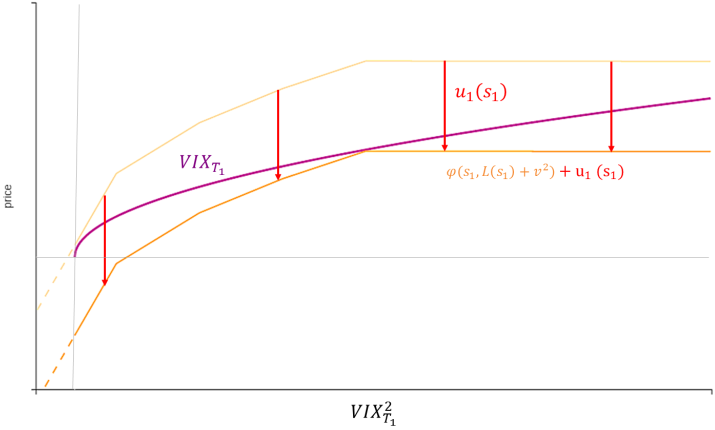
$$\forall s_1 > 0,\ \forall v \geq 0, \qquad u_1(s_1) + \varphi(s_1, L(s_1) + v^2) \leq v$$

Proposition

Let $\varphi : \mathbb{R}_+^* \times \mathbb{R} \to \mathbb{R}$ be a **concave** *function. The following portfolio* **subreplicates** *the VIX:*

$$\begin{aligned} u_1(s_1) &= \varphi_{\text{sub}}^*(s_1), & u_2(s_2) &= \varphi(s_2, L(s_2)) \\ \Delta^S(s_1, v) &= -\partial_{1,r}\varphi(s_1, L(s_1) + v^2), & \Delta^L(s_1, v) &= -\partial_{2,r}\varphi(s_1, L(s_1) + v^2) \end{aligned}$$

A new family of functionally generated sub/superreplicating portfolios

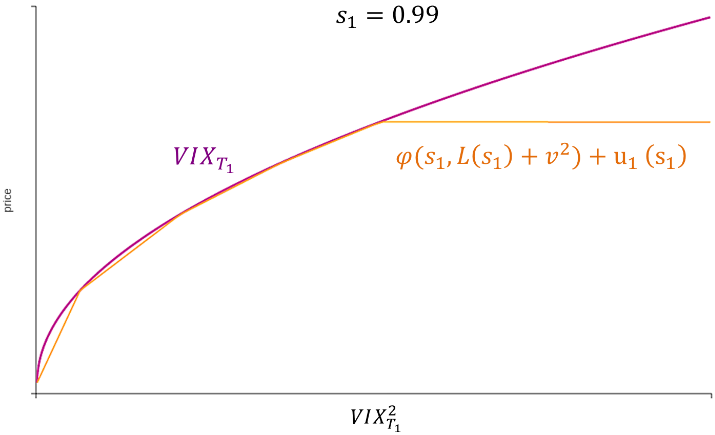


A new family of functionally generated sub/superreplicating portfolios

- Portfolio family parameterized by convex/concave function $\varphi : \mathbb{R}_+^* \times \mathbb{R} \to \mathbb{R}$
- $\Longrightarrow$ Optimize over convex/concave functions in dimension 2
- Problem: How to test all convex/concave functions in dimension 2?
- Dimension 1: Extreme rays of the convex cone of convex functions = call/put payoffs
- Johansen (1972): In dimension $d \geq 2$, the extreme rays are dense in the cone
- Scarsini [Multivariate convex orderings, dependence, and stochastic equality, J. Appl. Prob., 35:93–103, 1998]: **"No simple characterization for the convex ordering in dimension $d \geq 2$ can be hoped for."**
- Workaround: Consider only the functions of the type $\varphi(x, y) = \psi(ax + y)$ with $\psi : \mathbb{R} \to \mathbb{R}$ convex/concave
- Optimize on a and ψ: decompose ψ on call/put payoffs, and optimize on weights

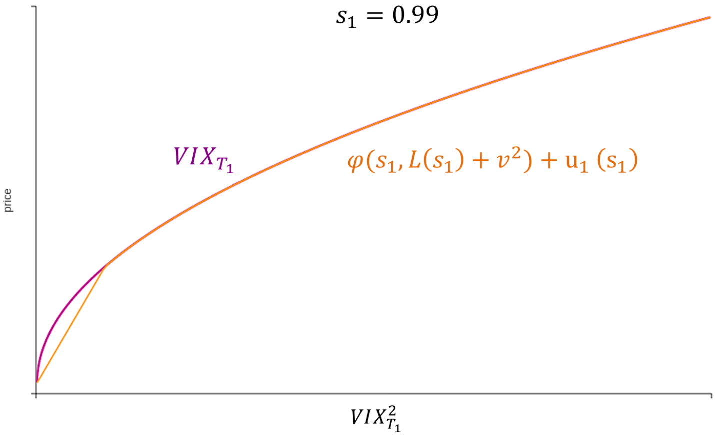
Examples of subreplicating portfolios

- $\varphi(x, y) = \psi(ax + y)$, ψ concave polygonal line (piecewise affine)



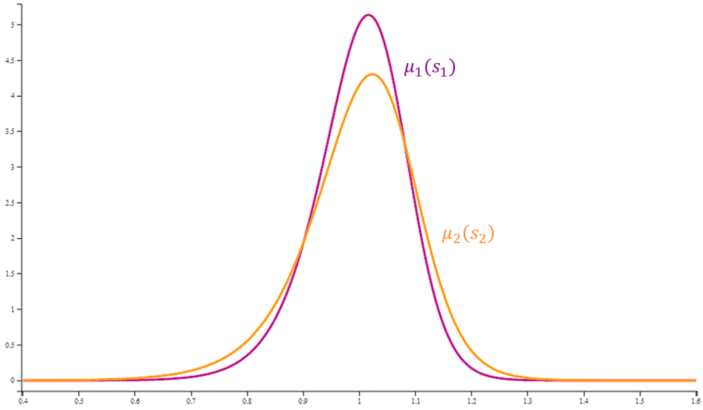
Examples of subreplicating portfolios

- $\varphi(x,y) = \psi(ax+y)$, ψ cut square root



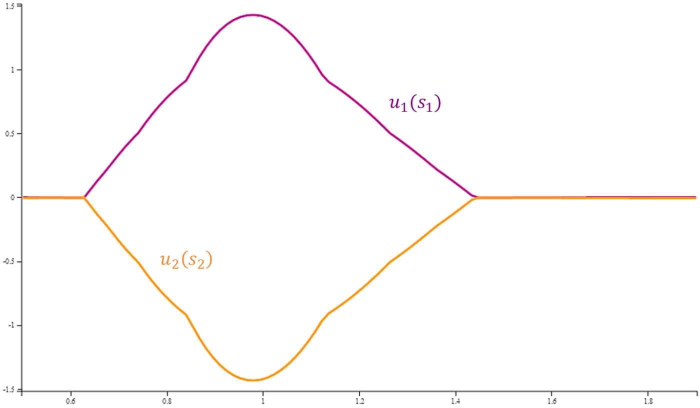
Numerical results

SABR risk-neutral densities



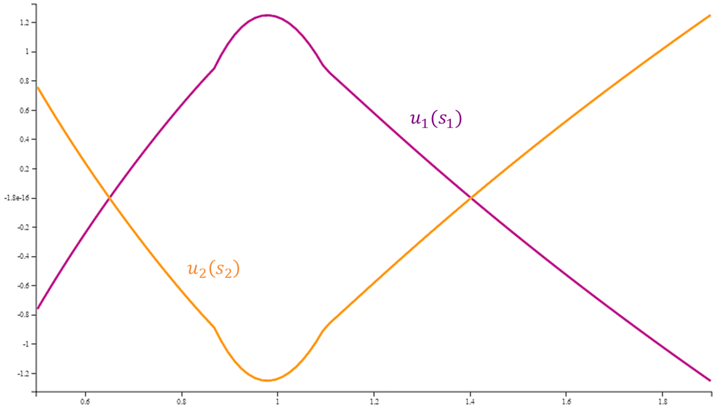
Numerical results

- $\varphi(x, y) = \psi(ax + y)$, ψ concave polygonal line (piecewise affine)



Numerical results

- $\varphi(x, y) =$ cut square root



Numerical results

		SABR model $T_1 = 2$ months	Market smiles as of May 5, 2016; $T_1 = 10$ days
Lower bound	Classical lower bound	0%	0%
	ψ polygonal ($N = 1$ kink)	4.6%	4.4%
	ψ polygonal ($N = 10$ kinks)	5.2%	7.2%
	ψ cut square root	6.0%	7.8%
	Lower bound from LP solver	7.2%	8.4%
Classical upper bound		22.8%	16.7%
Upper bound from LP solver		22.8%	16.7%

- Numerically: upper bound $= \sigma_{12}$
- Theoretically: Can we characterize the market smiles μ_1 and μ_2 for which optimal upper bound $= \sigma_{12}$?

3. When are the classical bounds optimal?

Monge-Kantorovich duality

- $\mathcal{M}(\mu_1, \mu_2)$ = the martingale measures μ on $(\mathbb{R}_+^*)^2$ with marginals μ_1 and μ_2:

$$S_1 \sim \mu_1, \qquad S_2 \sim \mu_2, \qquad \mathbb{E}^\mu [S_2 | S_1] = S_1$$

- Price at T_1 of the log-contract in Model $\mu \in \mathcal{M}(\mu_1, \mu_2)$:

$$\Lambda_\mu(S_1) \equiv \mathbb{E}^\mu \left[L\left(\frac{S_2}{S_1}\right) \middle| S_1 \right]$$

Corresponds to the situation where the **VIX is a function of S_1**

- For all $\mu \in \mathcal{M}(\mu_1, \mu_2)$,

$$\begin{aligned} \Lambda_\mu(S_1) &\geq 0 \\ \mathbb{E}^1[\Lambda_\mu(S_1)] &= \sigma_{12}^2 \end{aligned}$$

Monge-Kantorovich duality

- $\mathcal{M}(\mu_1, \mu_2)$ naturally arises in the dual formulation of the classical martingale optimal transportation problem (no VIX):

$$
\begin{aligned}
(u_1, u_2, \Delta) \in \mathcal{U}_{\text{super}} \quad &\Longleftrightarrow \quad u_1(s_1) + u_2(s_2) + \Delta(s_1)(s_2 - s_1) \geq g(s_1, s_2) \\
P_{\text{super}} \quad &\equiv \quad \inf_{\mathcal{U}_{\text{super}}} \left\{ \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^2[u_2(S_2)] \right\} \\
D_{\text{super}} \quad &\equiv \quad \sup_{\mu \in \mathcal{M}(\mu_1, \mu_2)} \mathbb{E}^\mu[g(S_1, S_2)]
\end{aligned}
$$

- MK duality: $D_{\text{super}} \leq P_{\text{super}}$ (weak), $D_{\text{super}} = P_{\text{super}}$ (strong)
- With the VIX:

$$
\begin{aligned}
(u_1, u_2, \Delta^S, \Delta^L) \in \mathcal{U}_{\text{super}} \quad &\Longleftrightarrow \quad u_1(s_1) + u_2(s_2) + \Delta^S(s_1, v)(s_2 - s_1) \\
&\qquad + \Delta^L(s_1, v)(L(s_2/s_1) - v^2) \geq v \\
P_{\text{super}} \quad &\equiv \quad \inf_{\mathcal{U}_{\text{super}}} \left\{ \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^2[u_2(S_2)] \right\} \\
D_{\text{super}} \quad &\equiv \quad \sup_{\mu \in \mathcal{M}_V(\mu_1, \mu_2)} \mathbb{E}^\mu[V]
\end{aligned}
$$

Monge-Kantorovich duality

- $\mathcal{M}_V(\mu_1, \mu_2)$ = the set of all the probability distributions μ on $(\mathbb{R}_+^*)^2 \times \mathbb{R}_+$ such that

$$S_1 \sim \mu_1, \quad S_2 \sim \mu_2, \quad \mathbb{E}^\mu \left[S_2 | S_1, V\right] = S_1, \quad \mathbb{E}^\mu \left[L\left(\frac{S_2}{S_1}\right) \middle| S_1, V \right] = V^2$$

  Corresponds to the general situation where the **VIX V is not necessarily a function of S_1**

- When we project V^2 onto functions of S_1, we must find back $\Lambda_\mu(S_1)$:

$$\mathbb{E}^\mu[V^2 | S_1] = \Lambda_\mu(S_1)$$

  (extending the definition of $\Lambda_\mu(S_1)$ to $\mu \in \mathcal{M}_V(\mu_1, \mu_2)$)

- Note that for all $\mu \in \mathcal{M}_V(\mu_1, \mu_2)$,

$$\mathbb{E}^\mu[V^2] = \sigma_{12}^2$$

Absence of a duality gap

Superreplication of $f(s_1, s_2, v)$, e.g., $f(s_1, s_2, v) = v$:

Theorem

Let $f : \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}$ *be upper semicontinuous and satisfy*

$$|f(s_1, s_2, v)| \leq C\big(1 + s_1 + s_2 + |L(s_1)| + |L(s_2)| + v^2\big)$$

for some constant $C > 0$. *Then*

$$D_{\text{super}} \equiv \sup_{\mu \in \mathcal{M}_V(\mu_1,\mu_2)} \mathbb{E}^{\mu}[f] = \inf_{\mathcal{U}_{\text{super}}} \left\{ \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^2[u_2(S_2)] \right\} \equiv P_{\text{super}}.$$

Moreover, $D_{\text{super}} \neq -\infty$ *if and only if* $\mathcal{M}_V(\mu_1, \mu_2) \neq \emptyset$*, and in that case the supremum is attained.*

Of course a similar results holds for the subreplication of $f(s_1, s_2, v)$

Local volatility property of the superreplication price

Superreplication of VIX futures: A dual optimal measure can be chosen of the "local volatility type", i.e., s.t. the VIX is a function of S_1:

Proposition

$$D_{\text{super}} \equiv \sup_{\mu \in \mathcal{M}_V(\mu_1,\mu_2)} \mathbb{E}^\mu[V] = \sup_{\mu \in \mathcal{M}(\mu_1,\mu_2)} \mathbb{E}^1\left[\sqrt{\Lambda_\mu(S_1)}\right]$$

What do we mean by arbitrage-free?

***S*-arbitrage**: when only the S&P 500 and vanilla options on it are available for trading:

- $\mathcal{U}_S^0 =$ the functions (u_1, u_2, Δ) s.t.

$$u_1(s_1) + u_2(s_2) + \Delta(s_1)(s_2 - s_1) \geq 0$$

- An arbitrage = an element of $\mathcal{U}_S^0$ such that $\mathbb{E}^1[u_1(S_1)] + \mathbb{E}^2[u_2(S_2)] < 0$

(S, V)-arbitrage: when it is also possible to trade the log-contract at T_1:

- $\mathcal{U}_{S,V}^0 =$ the functions $(u_1, u_2, \Delta^S, \Delta^L)$ s.t.

$$u_1(s_1) + u_2(s_2) + \Delta^S(s_1, v)(s_2 - s_1) + \Delta^L(s_1, v)\left(L\left(\frac{s_2}{s_1}\right) - v^2\right) \geq 0$$

- An arbitrage = an element of $\mathcal{U}_{S,V}^0$ such that $\mathbb{E}^1[u_1(S_1)] + \mathbb{E}^2[u_2(S_2)] < 0$

What do we mean by arbitrage-free?

Clearly, absence of (S, V)-arbitrage implies absence of S-arbitrage. Actually, both notions are equivalent:

Theorem

The following assertions are equivalent:

(i) *The market is free of S-arbitrage,*

(ii) *The market is free of (S, V)-arbitrage,*

(iii) $\mathcal{M}(\mu_1, \mu_2) \neq \emptyset$,

(iv) $\mathcal{M}_V(\mu_1, \mu_2) \neq \emptyset$,

(v) μ_1 *and* μ_2 *are in convex order, i.e., for any convex function* $f : \mathbb{R}_+^* \to \mathbb{R}$, $\mathbb{E}^1[f(S_1)] \leq \mathbb{E}^2[f(S_2)]$.

- Not surprisingly, the possibility of trading the FSLC at T_1 does not add any restriction in our model-free setting
- However, our proof needs the difficult direction of Strassen's theorem

When is the classical upper bound optimal?

Denote

$$\ell_1 \equiv \mathbb{E}^1[L(S_1)], \qquad \ell_2 \equiv \mathbb{E}^2[L(S_2)]; \qquad \sigma_{12}^2 = \ell_2 - \ell_1$$

Let $\bar{\mathcal{M}}(\mu_1, \mu_2)$ be the subset of $\mathcal{M}(\mu_1, \mu_2)$ made of the martingale measures μ such that $\Lambda_\mu(S_1)$ is a.s. constant. In this case, $\Lambda_\mu(S_1) = \sigma_{12}^2$ a.s.

Theorem

The following assertions are equivalent:

(i) $P_{\text{super}} = \sigma_{12}$,

(ii) *there exists* $\mu \in \mathcal{M}_V(\mu_1, \mu_2)$ *such that* $V = \sigma_{12}$ μ*-a.s.,*

(iii) $\bar{\mathcal{M}}(\mu_1, \mu_2) \neq \emptyset$,

(iv) $\mathrm{Law}_{\mu_1}(S_1, L(S_1) - \ell_1)$ *and* $\mathrm{Law}_{\mu_2}(S_2, L(S_2) - \ell_2)$ *are in convex order,*

(iv') $\mathrm{Law}_{\mu_1}(S_1, L(S_1) + \sigma_{12}^2)$ *and* $\mathrm{Law}_{\mu_2}(S_2, L(S_2))$ *are in convex order.*

A condition under which the classical upper bound is not optimal

- Necessary and sufficient conditions in the previous theorem are not straightforward to check given the marginals: no simple family of convex test functions exists in two or more dimensions
- $\Longrightarrow$ We are interested in simpler criteria, at the expense of not being sharp. The following is a condition that involves only call and put prices:

Proposition

Denote by $C_i(K)$ and $P_i(K)$ the prices at time 0 of the call and put options with maturity T_i and strike K. Let

$$\Psi_1(K) \equiv \Phi_1\left(K + \frac{1}{2}\sigma_{12}^2\tau\right), \quad \Psi_2(K) \equiv \Phi_2(K),$$

where

$$\Phi_i(K) = \begin{cases} \frac{C_i(e^K)}{S_0} - \int_{e^K}^{\infty} \frac{C_i(k)}{k^2}dk & \text{if } K > \ln S_0 \\ \ln S_0 - K + \frac{P_i(e^K)}{S_0} - \int_{e^K}^{S_0} \frac{P_i(k)}{k^2}dk - \int_{S_0}^{\infty} \frac{C_i(k)}{k^2}dk & \text{otherwise.} \end{cases}$$

If there exists $K \in \mathbb{R}$ such that $\Psi_1(K) > \Psi_2(K)$, then $P_{\mathrm{super}} < \sigma_{12}$.

When is the classical lower bound optimal?

The classical lower bound is never sharp in practice:

Theorem

$P_{\text{sub}} = 0$ *if and only if* $\mu_1 = \mu_2$.

Better: when $\mu_1 \neq \mu_2$, we can construct an explicit, functionally generated subreplicating portfolio that has strictly positive price:

Proposition

Let $\mu_1 \neq \mu_2$ *be in convex order. Then there exists a functionally generated subreplicating portfolio* $(u_1, u_2, \Delta^S, \Delta^L) \in \mathcal{U}_{\text{sub}}$ *with strictly positive price. It is generated by the concave function* $\psi(z) \equiv -\gamma(z+b)_-$ *and a constant* $a > 0$, *where* $\gamma > 0$ *and* $b \in \mathbb{R}$. *The values of the constants depend on* μ_1, μ_2 *and can be found explicitly.*

4. Specific families of smiles μ_1, μ_2 and corresponding portfolios

Examples where $P_{\text{super}} < \sigma_{12}$

Optimal upper bound P_{super} < classical upper bound σ_{12}:

- when μ_2 = a Bernoulli distribution:
  - except for very special μ_1
- when μ_2 has compact support:
  - if μ_1 puts weight close to the edges of $\text{supp}(\mu_2)$
- when μ_2 = a three-point distribution:
  - if and only if a very graphical condition on $\text{supp}(\mu_1)$ holds

The case where μ_2 is a Bernoulli distribution

$$\mu_2 = p\delta_{s_2^u} + (1-p)\delta_{s_2^d}, \qquad 0 < s_2^d < S_0 < s_2^u, \qquad p = \frac{S_0 - s_2^d}{s_2^u - s_2^d} \in (0,1) \quad (5.1)$$

- In the absence of arbitrage, the sets $\mathcal{M}(\mu_1, \mu_2)$ and $\mathcal{M}_V(\mu_1, \mu_2)$ both have a unique element (**complete model**)
- $\Longrightarrow D_{\text{super}} = D_{\text{sub}}$ and this number can be computed explicitly as the expectation under the unique risk-neutral measure
- Unique martingale transition probabilities:

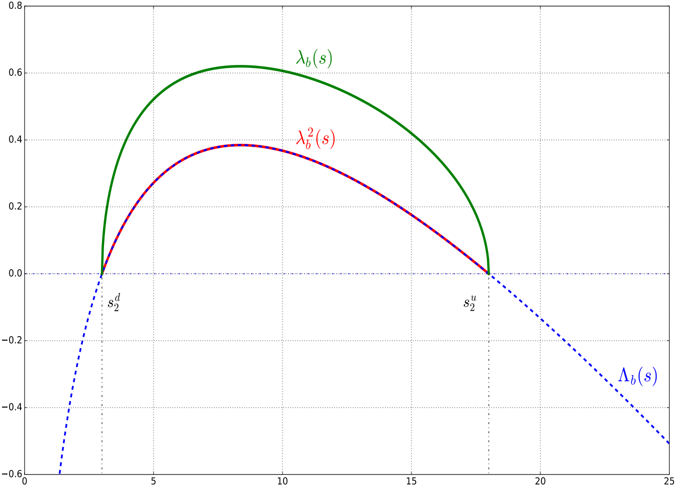
$$\pi_u(s_1) \equiv \frac{s_1 - s_2^d}{s_2^u - s_2^d}, \qquad \pi_d(s_1) \equiv \frac{s_2^u - s_1}{s_2^u - s_2^d} = 1 - \pi_u(s_1)$$

- Risk-neutral price of the FSLC in the one-step binomial model:

$$\Lambda_b(s_1) \equiv \pi_u(s_1) L\left(\frac{s_2^u}{s_1}\right) + \pi_d(s_1) L\left(\frac{s_2^d}{s_1}\right), \qquad s_1 > 0$$

- Its square root: $\lambda_b(s_1) = \sqrt{\Lambda_b(s_1)}, \quad s_1 \in [s_2^d, s_2^u]$

The case where μ_2 is a Bernoulli distribution



The case where μ_2 is a Bernoulli distribution

Theorem

Let μ_2 be the Bernoulli distribution (5.1). Then, there is no arbitrage, or equivalently $\mathcal{M}(\mu_1, \mu_2) \neq \emptyset$, if and only if $\mathrm{supp}(\mu_1) \subset [s_2^d, s_2^u]$. In this case, $\mathcal{M}(\mu_1, \mu_2)$ has a unique element β, given by

$$\beta(ds_1, ds_2) \quad = \quad \mu_1(ds_1)\left(\pi_u(s_1)\delta_{s_2^u}(ds_2) + \pi_d(s_1)\delta_{s_2^d}(ds_2)\right)$$

and $\mathcal{M}_V(\mu_1, \mu_2)$ has a unique element β_V, given by

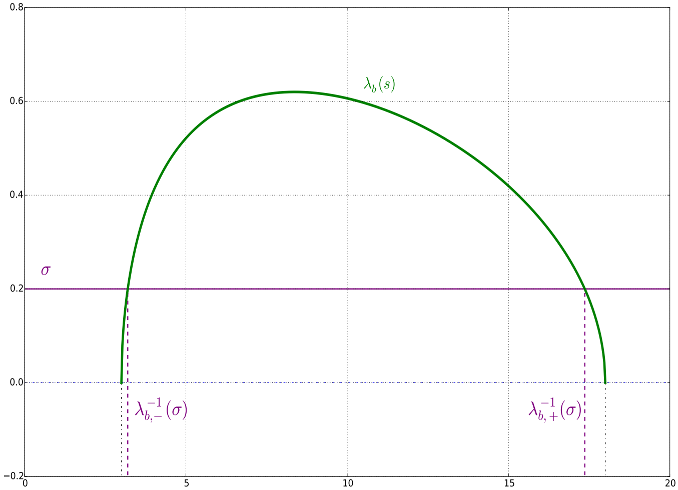
$$\beta_V(ds_1, ds_2, dv) = \beta(ds_1, ds_2)\delta_{\lambda_b(s_1)}(dv).$$

In particular, $V = \lambda_b(S_1)$ β_V-a.s. Moreover,

(i) *if μ_1 is the Bernoulli distribution that takes values in $\{\lambda_{b,-}^{-1}(\sigma), \lambda_{b,+}^{-1}(\sigma)\}$ for some $\sigma \in [0, \lambda_b(S_0)]$, then $\beta \in \bar{\mathcal{M}}(\mu_1, \mu_2)$ and $P_{\mathrm{super}} = \sigma_{12}$,*

(ii) *if μ_1 is a different distribution, then $\bar{\mathcal{M}}(\mu_1, \mu_2) = \emptyset$ and $P_{\mathrm{super}} = \mathbb{E}^1[\lambda_b(S_1)] < \sigma_{12}$.*

The case where μ_2 is a Bernoulli distribution

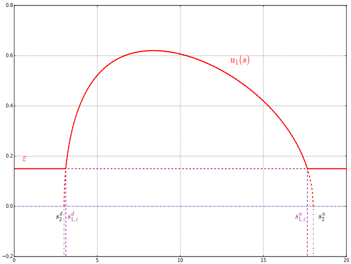


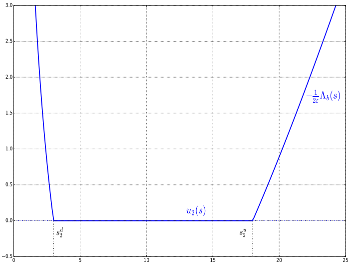
The case where μ_2 is a Bernoulli distribution

- Back to the primal problem: Find ε-optimal portfolio u_1, u_2, Δ^S, Δ^L
- Such a portfolio can be chosen of the functionally generated form $\varphi(x,y) = \psi(ax+y)$ with $\psi(z) \equiv \frac{1}{2\varepsilon}(z+b)_+$:

$$u_1(s_1) = \begin{cases} \lambda_b(s_1) & \text{if } s_1 \in [s^d_{1,\varepsilon}, s^u_{1,\varepsilon}] \\ \varepsilon & \text{if } s_1 \notin [s^d_{1,\varepsilon}, s^u_{1,\varepsilon}], \end{cases} \qquad \Delta^S(s_1, v) = \frac{\Delta_b}{2\varepsilon} \mathbf{1}_{v \geq \lambda_b(s_1)},$$

$$u_2(s_2) = \begin{cases} 0 & \text{if } s_2 \in [s^d_2, s^u_2] \\ -\frac{1}{2\varepsilon}\Lambda_b(s_2) & \text{if } s_2 \notin [s^d_2, s^u_2], \end{cases} \qquad \Delta^L(s_1, v) = -\frac{1}{2\varepsilon} \mathbf{1}_{v \geq \lambda_b(s_1)}$$





The case where μ_2 has compact support in $\mathbb{R}_+^*$

$$s_2^d \equiv \min \operatorname{supp}(\mu_2) > 0, \qquad s_2^u \equiv \max \operatorname{supp}(\mu_2) < +\infty \tag{5.2}$$

- We assume that the market is arbitrage-free, i.e., that μ_1 and μ_2 are in convex order
- $\Longrightarrow \operatorname{supp}(\mu_1) \subset [s_2^d, s_2^u]$; in particular, $s_2^d \leq s_1^d \leq s_1^u \leq s_2^u$, where

$$s_1^d \equiv \min \operatorname{supp}(\mu_1), \qquad s_1^u \equiv \max \operatorname{supp}(\mu_1)$$

Sufficient conditions on the market smiles μ_1 and μ_2 under which $P_{\text{super}} < \sigma_{12}$? 3 strategies:

- Find a superreplicating portfolio whose price is strictly smaller than σ_{12}; or
- Show that $\mathrm{Law}_{\mu_1}(S_1, L(S_1) - \ell_1)$ and $\mathrm{Law}_{\mu_2}(S_2, L(S_2) - \ell_2)$ are not in convex order; or
- Verify that $\bar{\mathcal{M}}(\mu_1, \mu_2) = \emptyset$.

The case where μ_2 has compact support in $\mathbb{R}_+^*$

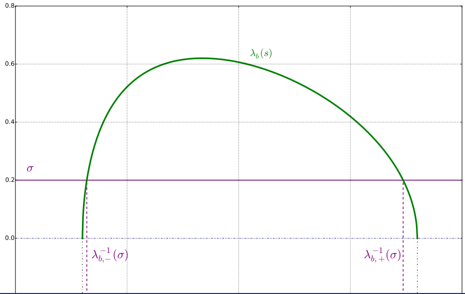
Proposition

Assume (5.2) and absence of arbitrage. Each of the following implies $P_{\text{super}} < \sigma_{12}$:

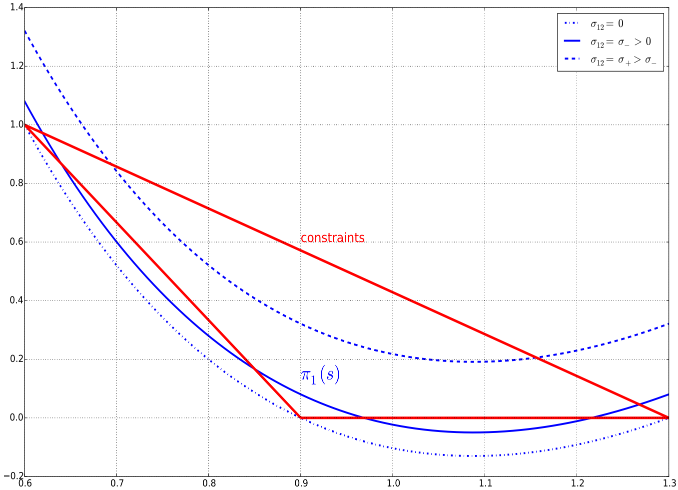
(i) $\mathbb{E}^1\left[\lambda_b(S_1)\right] < \sigma_{12}$,

(ii) $L(s_1^d) - \ell_1 > L(s_2^d) - \ell_2$, *which holds in particular if* $s_1^d = s_2^d$ *and* $\mu_1 \neq \mu_2$,

(iii) $\mu_1(A) > 0$ *for* $A \equiv (s_2^d, \lambda_{b,-}^{-1}(\sigma_{12})) \cup (\lambda_{b,+}^{-1}(\sigma_{12}), s_2^u)$.



The case where μ_2 is a three-point distribution



5. Conclusion

Conclusion: main takeaways

A longstanding, important question in volatility markets:

- What information on the dynamics of the surface of implied vol is contained in the surface itself?
- How is the surface dynamics constrained by the state of the surface itself?
- What information on the vol of vol is contained in the surface of implied vol?
- What information on the price of volatility derivatives is contained in the surface of implied vol?

Conclusion: main takeaways

Our contributions:

- Given two slices of the implied vol surface: the smiles at T_1 and T_2
- Volatility derivative: VIX future. Derivative on options-implied volatility
- **The price of the VIX future ($\sim$ stdev of VIX_{T_1}) is constrained**:
  - Well known: price cannot be too large (classical upper bound, corresponds to zero stdev of VIX_{T_1})
  - New: price cannot be too small, i.e., **stdev of VIX_{T_1} cannot be too large**

$$0\% \leq \text{price} \leq 16\% \longrightarrow 8\% \leq \text{price} \leq 16\%$$

  - New: **sharp bounds and the corresponding portfolios**, using
    - **LP solver** or
    - a **new family of functionally generated sub/superreplicating portfolios**
  - New: **in typical markets, classical upper bound is sharp: stdev of VIX_{T_1} can be zero**
  - New: explicit **examples where the classical upper bound is not sharp** and the corresponding portfolios
  - New: **absence of a duality gap** $\Longrightarrow$ optimal bounds can be computed in the dual manner

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