Local volatility models enhanced with jumps

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Joint work with Pierre Henry-Labordère [1]

[1] http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2781102

Mathematical analysis/algorithm

- Two models:
  - LVM with jumps:

$$\frac{dS_t}{S_t} = \sigma(t, S_t)\, dW_t + (J - 1)(dN_t - \lambda_t dt)$$

  - Regime-switching model:

$$\begin{aligned} \frac{dS_t}{S_t} &= \sigma_1(t, S_t)\, dW_t + (J - 1)(dN_t - \lambda_t dt), \quad \forall t \le \tau \\ \frac{dS_t}{S_t} &= \sigma_2(t, S_t)\, dW_t, \quad \forall t > \tau, \quad \sigma_2 := J_S \sigma_1 \end{aligned}$$

- Find σ so that: $\mathbb{E}(S_T - K)_+ = C^{\mathrm{mkt}}(T, K)$
- Leads to nonlinear McKean SDEs

Calibration condition-McKean SDE

- Dynamics of LVM with jumps:

$$\frac{dS_t}{S_t} = \sigma\left(t, S_t\right) dW_t + \left(J(S_t) - 1\right)\left(dN_t - \lambda_t dt\right)$$

- If $S_t \sim \mathbb{P}_t^{\mathrm{mkt}}$ then

$$\sigma^2(t, K) = \sigma^2_{\mathrm{Dupire}}(t, K) - 2\frac{\Lambda(t, K)}{K^2 \partial_{KK} C^{\mathrm{mkt}}(t, K)}.$$

where:

$$\Lambda(t, K) \equiv \mathbb{E}\left[\lambda_t(K - J(S_t)S_t)\left(1_{S_t > K} - 1_{J(S_t)S_t > K}\right)\right]$$

- No existence/unicity result for this McKean SDE.

McKean SDE: Regularizing the SDE

- Regularize the function $\sigma \to \sigma^{\epsilon}$.

Proposition 1

Let us define

$$\frac{dS_t^{\epsilon}}{S_t^{\epsilon}} = \sigma^{\epsilon}\left(t, S_t^{\epsilon}\right) dW_t + \left(J(S_t^{\epsilon}) - 1\right)\left(dN_t - \lambda_t dt\right)$$

Then this non-linear SDE admits a unique solution S_t^{ϵ}.

- How close is $\mathbb{E}(S_T^{\epsilon} - K)_+$ from $C^{\mathrm{mkt}}(T, K)$

Main theorem[2]

Theorem 2

For all $(t, K) \in [0, T] \times \mathbb{R}_+$*:*

$$\lim_{\epsilon \to 0} \mathbb{E}[(S_t^\epsilon - K)^+] = C^{\mathrm{mkt}}(t, K).$$

Proof:

$u_\epsilon(t, K) = \mathbb{E}[(S_t^\epsilon - K)^+] - C^{\mathrm{mkt}}(t, K)$. Find a PDE family (E_ϵ) s.t:

(i) u_ϵ solves (E_ϵ).

(ii) (E_ϵ) admits only one solution that tends to 0 as $\epsilon \to 0$.

[2] http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2781102

Numerical computations: particle method

N independent particles $(S^i, \lambda^i, N^i)_{i=1,\dots,N}$

Algorithm

$[0, T] = \bigcup_i [i\Delta, (i+1)\Delta]$.

1. Set $\sigma(t, S) = \sigma_{\mathrm{Dupire}}(t, S), \forall t \in [0, \Delta]$
2. Compute $\Lambda(\Delta, K)$:

$$\Lambda(\Delta, K) = \frac{1}{N} \sum_{i=1}^{N} \lambda^i_\Delta (K - JS^i_\Delta)(1_{S^i_\Delta > K} - 1_{S^i_\Delta J(S^i_\Delta) > K})$$

   set $\sigma(t, K) = \sigma(\Delta, K)\ \forall t \in [\Delta, 2\Delta]$.

3. Iterate up to T.

Numerical implementation

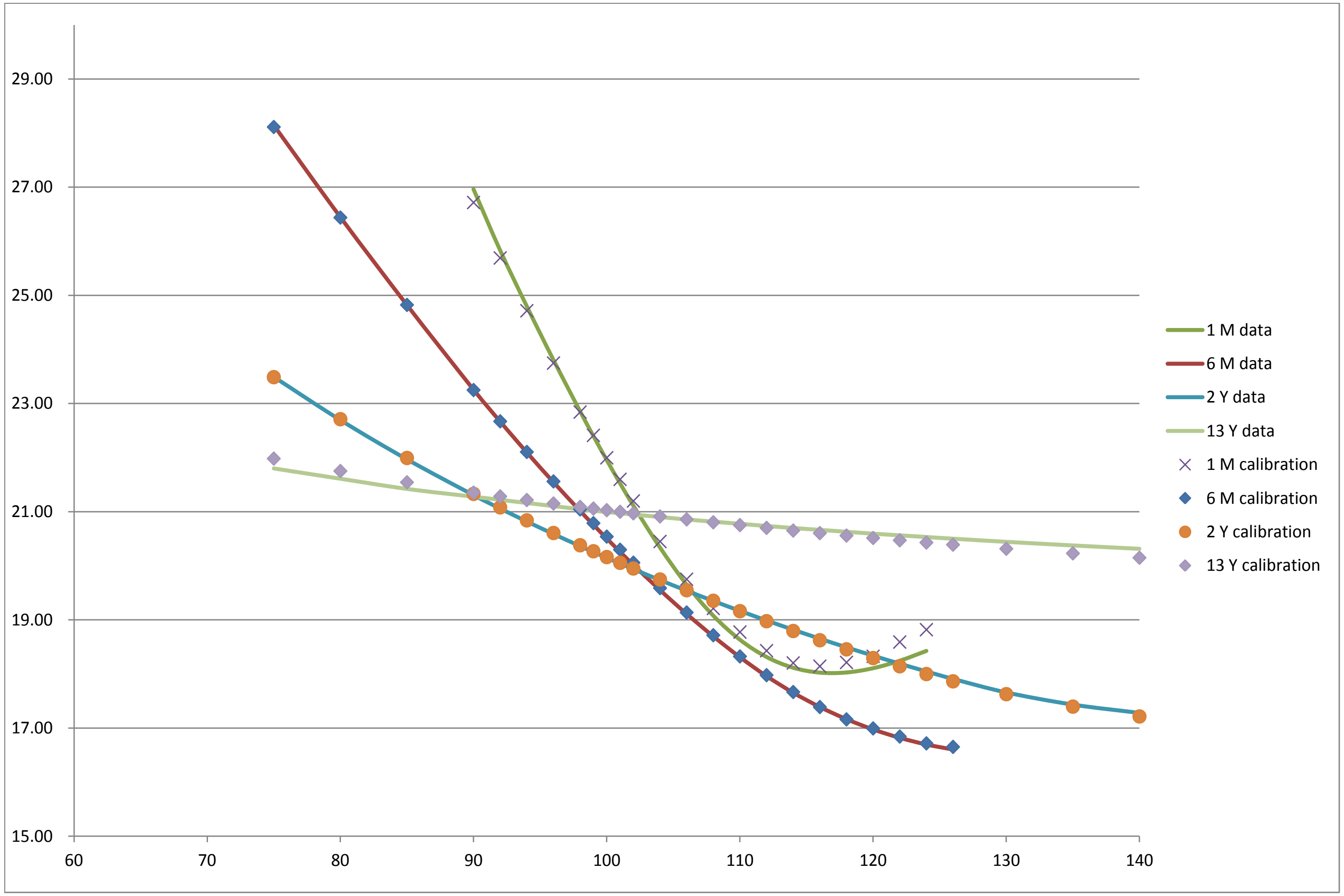

Figure: Calibrated implied volatilities compared to implied volatilities. $J = 0.9$, $\sigma_\lambda = 100\%$, $\rho_{S\lambda} = -40\%$, 10000 particles for the calibration.

Regime-switching model

SDE:

$$\begin{aligned}\frac{dS_t}{S_t} &= \sigma_1(t, S_t)\, dW_t + (J-1)(dN_t - \lambda_t dt), \quad \forall t \leq \tau \\ \frac{dS_t}{S_t} &= \sigma_2(t, S_t)\, dW_t, \quad \forall t > \tau\end{aligned}$$

where τ is the first time to default of a Poisson process $(N_t)_{t\geq 0}$.

Calibration condition

Setting $\sigma_2(t,K) = J_S(t,K)\sigma_1(t,K)$,

Proposition 3

$S_t \sim \mathbb{P}_t^{\mathrm{mkt}}$ for all $t \leq T$ if and only if

$$\begin{aligned} \sigma_1(t,K)^2 &= \frac{\sigma_{\mathrm{Dupire}}(t,K)^2}{1+(J_S(t,K)^2-1)P_2(t,K)} \\ &+ 2\frac{(J-1)(\Lambda(t,K)-K\partial_K\Lambda(t,K))}{\partial_{KK}C^{\mathrm{mkt}}(t,K)(1+(J_S(t,K)^2-1)P_2(t,K))} \\ &- 2\frac{J\Lambda(t,\frac{K}{J})-\Lambda(t,K)}{\partial_{KK}C^{\mathrm{mkt}}(t,K)(1+(J_S(t,K)^2-1)P_2(t,K))} \end{aligned}$$

where: $P_2(t,K) := \mathbb{E}[\mathbf{1}_{\tau<t}|S_t = K]$ and $\Lambda(t,K) := \mathbb{E}\left[\lambda_t \mathbf{1}_{\tau>t}(S_t-K)_+\right]$.

Numerical implementation

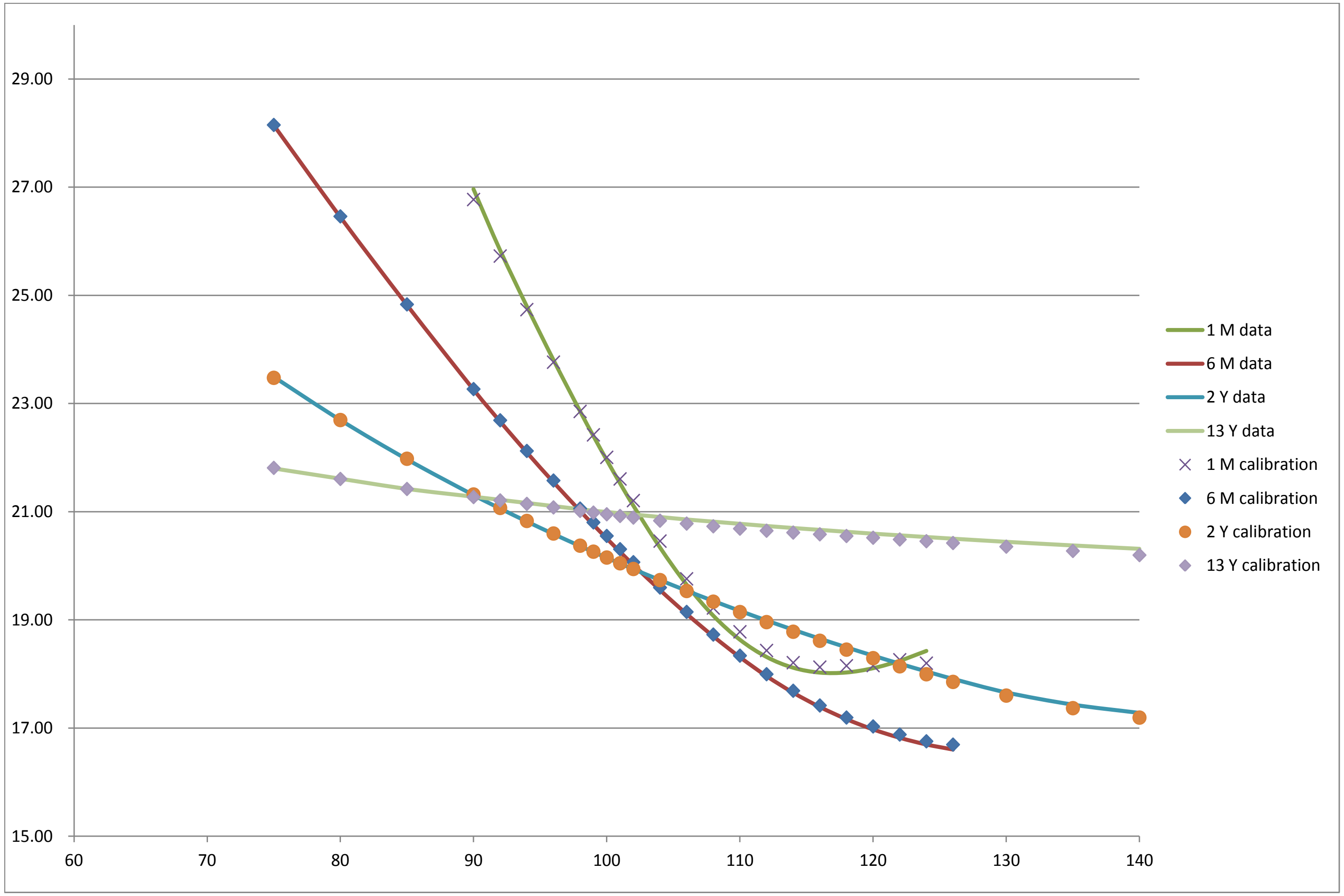

Figure: Calibrated implied volatilities compared to implied volatilities, $J = 0.9$, $J_S = 1.2$, $\sigma_\lambda = 100\%$, $\rho_{S\lambda} = -40\%$, 10000 particles for calibration.

Conclusion

- Robust numerical method
- The approach to prove: $\lim_{\epsilon \to 0} \mathbb{E}[(S_t^\epsilon - K)^+] = C^{\text{mkt}}(t, K)$ could be used for other calibration problems!