Parametric Option Pricing by Interpolation

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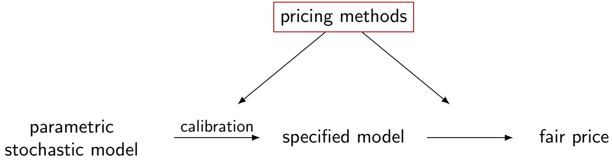
Introduction

- How to compute real-time prices in complex models?
- A new perspective: Parametric Option Pricing
- Enables complexity reduction here: by interpolation
- Error analysis
- Numerical Experiments



Complexity reduction to enable realistic stochastic models for finance.

Open Research Question

Jim Gatheral

”The rBergomi model captures the implied vol surface extremely well!”

Question: How can we **compute option prices** in the model?

Christian Bayer

”Pricing requires involved Monte Carlo simulation here. I need 20 minutes for one price.”

... not yet practicable

Parametric Option Pricing = POP

If you are interested in the a version of the slides including the photos please send me an email: kathrin.glau@tum.de

Parametric Option Pricing (POP)

Computation of $\mathrm{Put}^{K,T,q} = E\big(max\{(K - S_T^q), 0\}\big)$

Traditional approach

1. **Characterize** distribution of S_T^q and payoff function
2. **Distill** pricing method (PDE, Monte Carlo, ...)

Disadvantage: Limited possibilities for speed-up

New Perspective

1. **Repeatedly** compute $\mathrm{Put}^{K,T,q}$ for all (K, T, q)
2. **Exploit** parametric dependency $(K, T, q) \mapsto \mathrm{Put}^{K,T,q}$

Advantage: Enables application of complexity reduction techniques

Methods for POP

Fast Fourier Transform

+ for plain vanilla options and a large number of strikes

− only one specific parameter

Carr and Madan (1999), Raible (2000),. . .

Reduced Basis

+ for different types of parameters

− bound to partial differential equations

Sachs and Schu (2010), Cont et al. (2011), Haasdonk et al. (2012)

It's magic

New method: Magic point integration

> *Magic Points in Finance: Empirical Interpolation for Parametric Option Pricing*, M. Gaß, K. Glau and M. Mair (2015), arxiv: 1511.00884
>
> *Parametric Integration by Magic Point Empirical Interpolation*, M. Gaß and K. Glau (2015), arxiv: 1511.08510

...
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Architecture of Offline/Online Procedure

offline:

- reading the complexity
- extracting the essential information

online:

- real-time evaluation for the parameters of interest

Key Idea

Exploit **functional dependence**

$$p = (K, T, q) \mapsto Price^{K,T,q} := E\big[f^K(S_T^q)\big],$$

where f^K is a payoff function with parameter K,

... by interpolation in the parameters!

Chebyshev Interpolation for POP

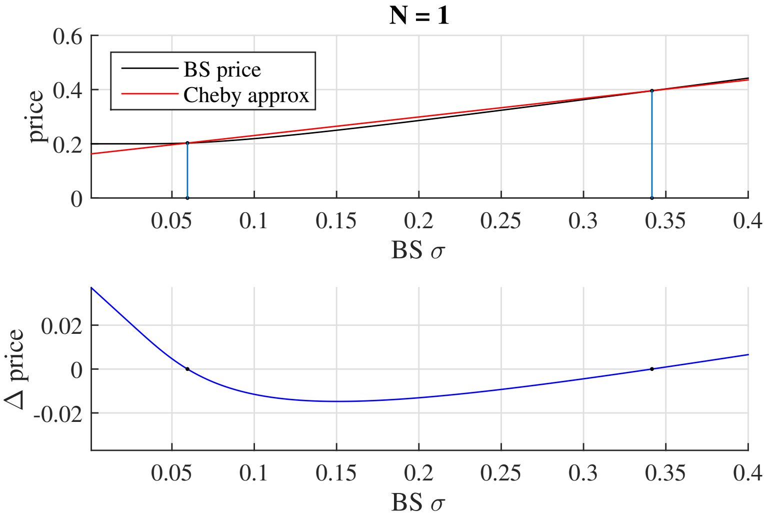
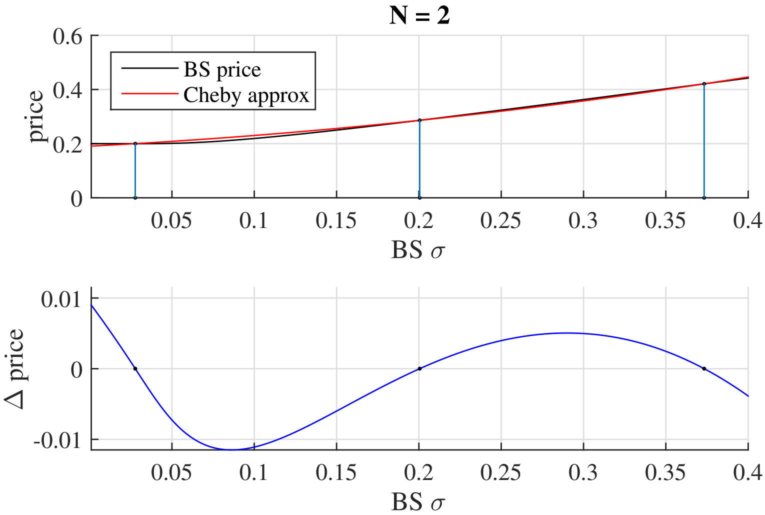


Figure : Call prices in the BS model

Chebyshev Interpolation for POP



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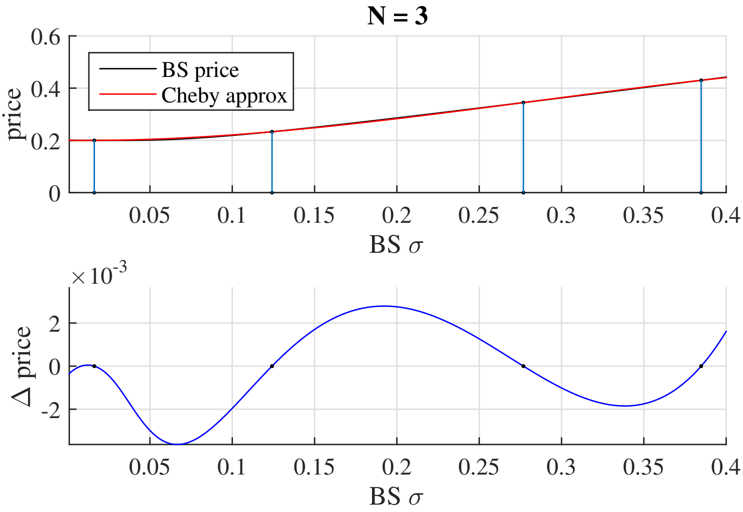

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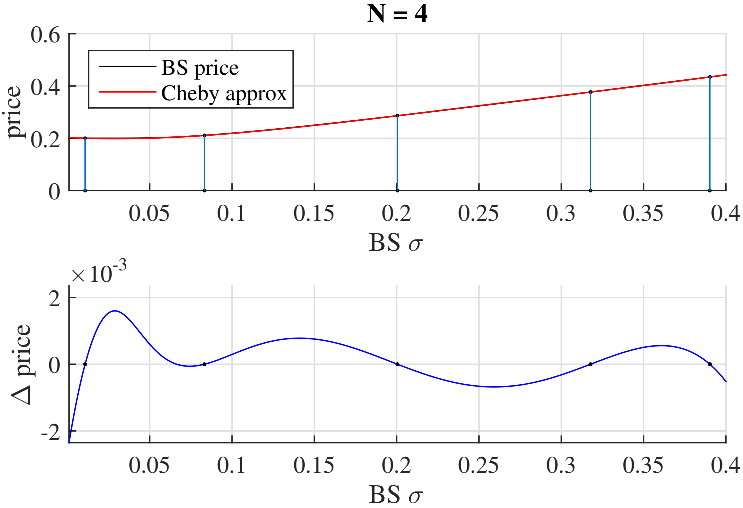
Chebyshev Interpolation for POP



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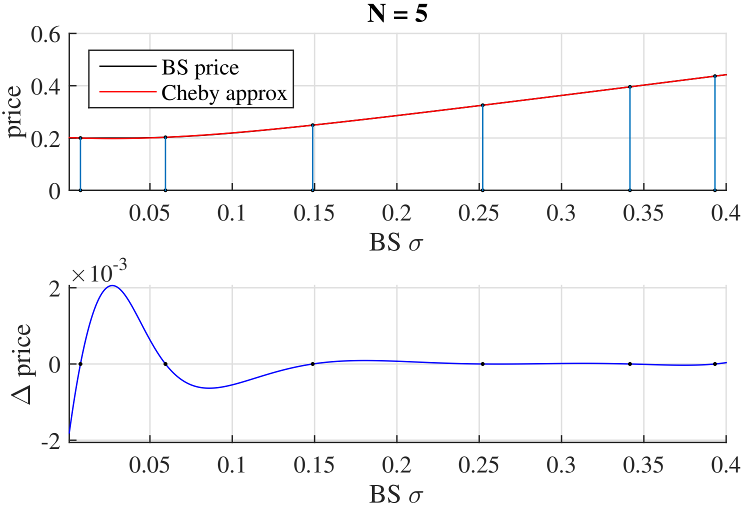
Chebyshev Interpolation for POP



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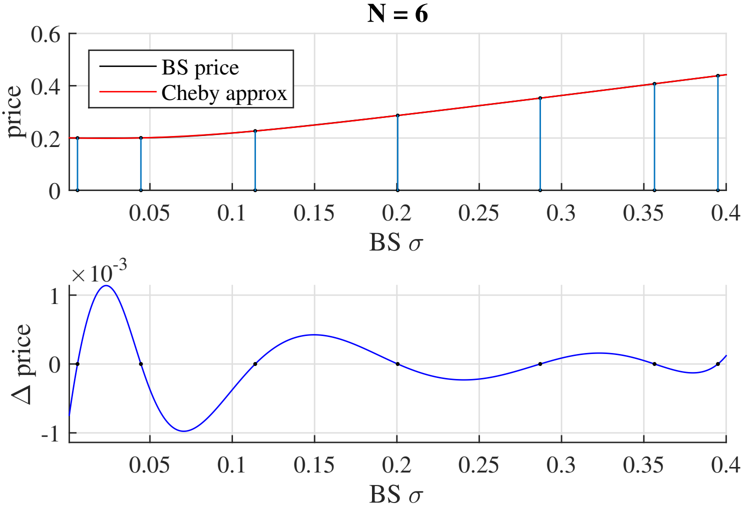

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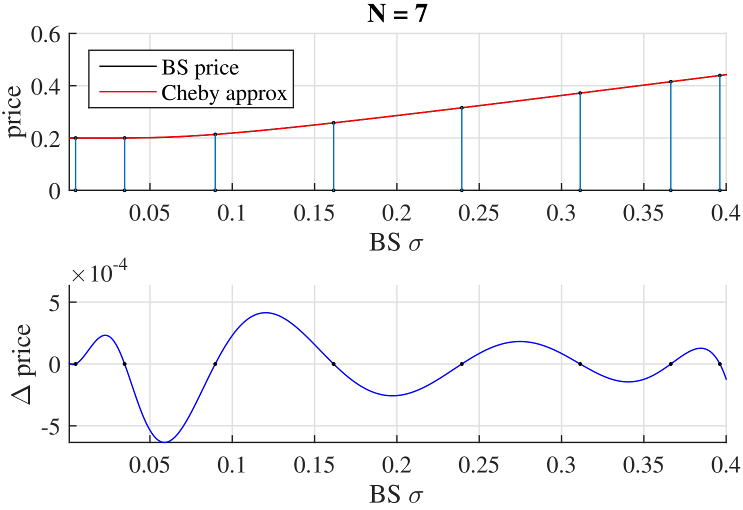

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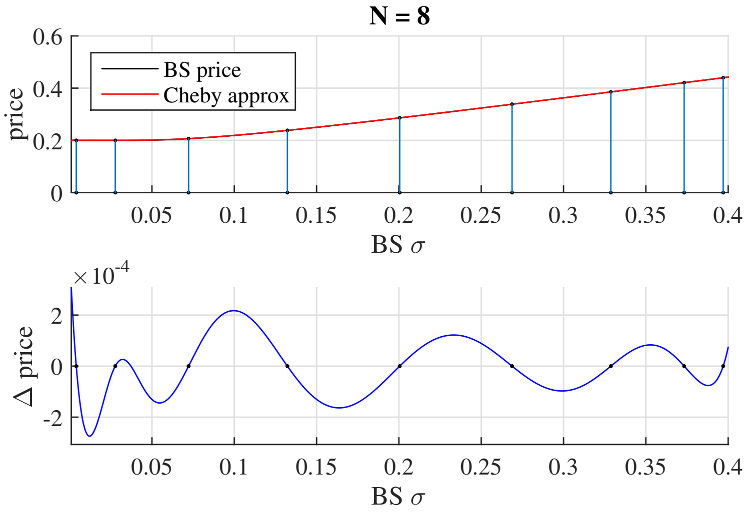

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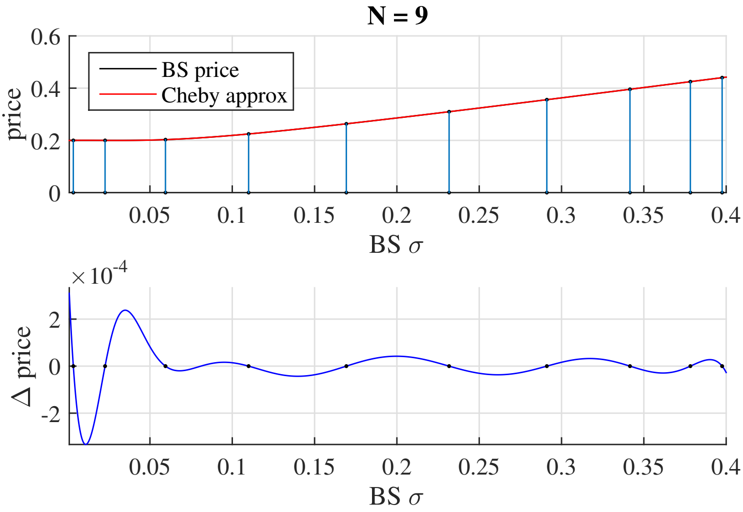

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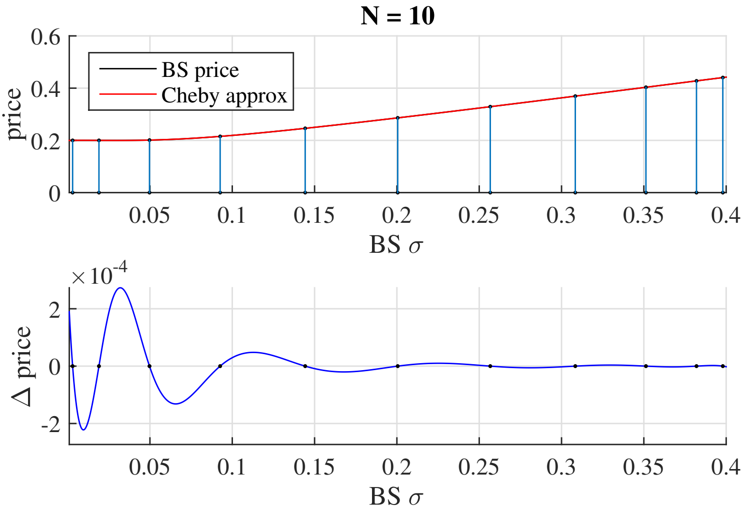

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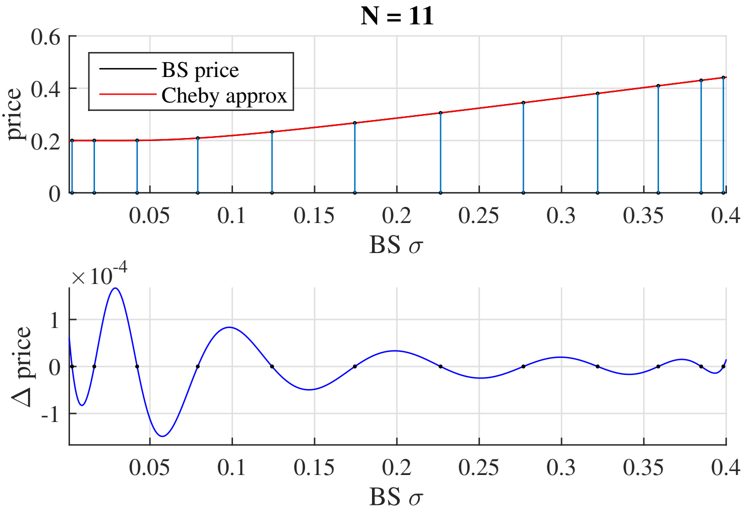
Chebyshev Interpolation for POP



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Chebyshev Interpolation for POP

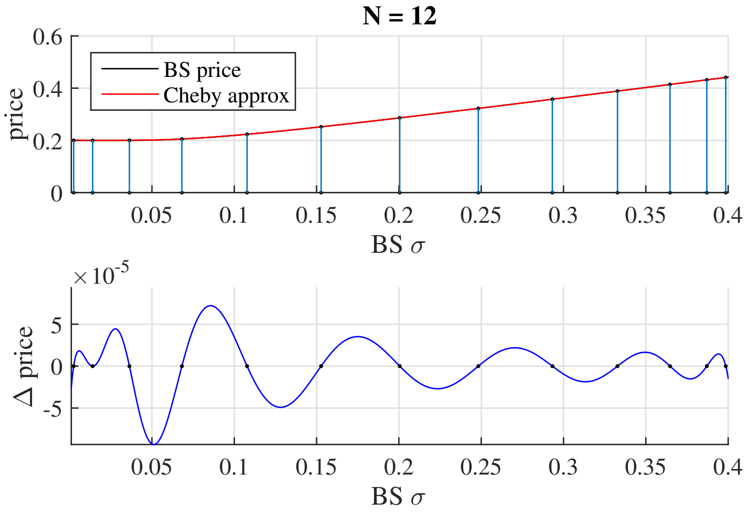
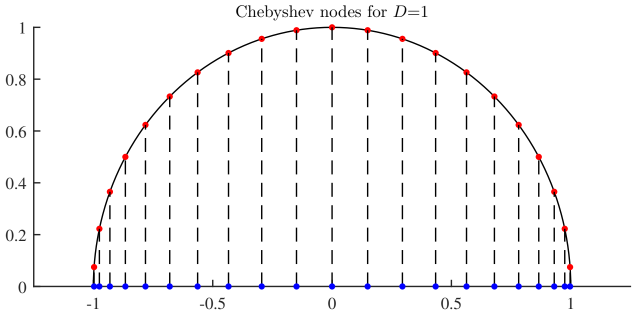


Figure : Call prices in the BS model

Chebyshev Interpolation

Chebyshev Interpolation Points



Chebyshev Polynomial Interpolation for POP

We approximate for log-strike $K = [-1, 1]$,

$$K \mapsto Price^K = E\left[f^K(S_T^q)\right]$$

by **Chebyshev polynomial interpolation**

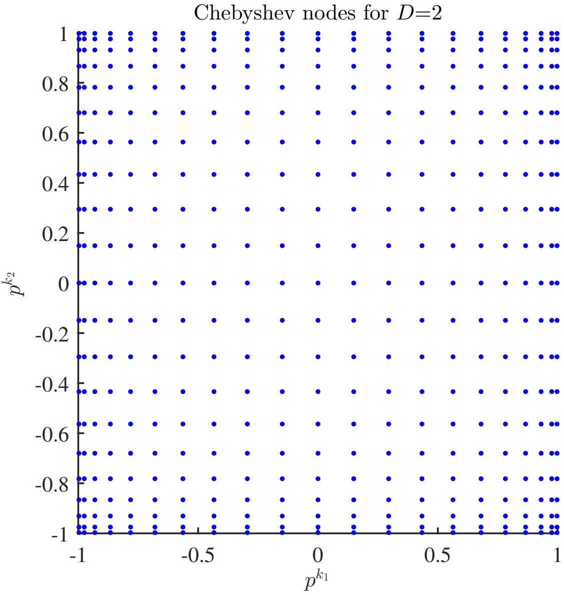
$$Price^K \approx 2 \sum_{0 \leq j \leq N} c_j(Price) T_j(K),$$

with

nodes	$p_k := \cos\left(\pi \frac{k}{N}\right)$
Chebyshev polynomials	$T_j(K) := \cos\left(j \arccos(K)\right)$
coeffients	$c_j(Price) := \sum''_{0 \leq k \leq N} Price^{p_k} \cos\left(j\pi \frac{k}{N}\right).$

$\sum''$ indicates that the first and last summand is multiplied by $1/2$.

Tensorized Chebyshev Interpolation



Tensorized Chebyshev Interpolation

Offline: For all Chebyshev nodes $p^{k_1,\cdots,k_D}$, compute the

- option prices

$$Price^{p^{(k_1,\ldots,k_D)}},$$

- coefficients

$$c_{(j_1,\ldots,j_D)} = \Big(\prod_{i=1}^{D} \frac{2^{\mathbb{1}_{\{j_i>0\}}}}{(N_i+1)}\Big) \sum_{k_1=0}^{N_1}{}'' \ldots \sum_{k_D=0}^{N_D}{}'' Price^{p^{(k_1,\ldots,k_D)}} \prod_{i=1}^{D} \cos\left(j_i \pi \frac{k_i}{N}\right).$$

Online:

- Given parameter p, evaluate the polynomial,

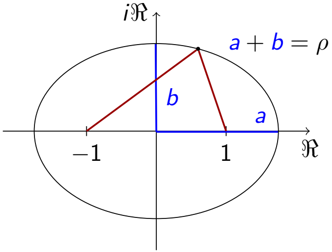
$$I_{N_1\ldots N_D}(Price)(p) := \sum_{j_1=0}^{N_1} \ldots \sum_{j_D=0}^{N_D} c_{(j_1,\ldots,j_D)} \prod_{i=1}^{D} T_{j_i}(p_i).$$

Convergence Analysis

Why Chebyshev Interpolation?

– Chebyshev interpolation of $f : [-1,1] \to \mathbb{R} \cong$ Fourier approximation of a periodic function.

– By implementing Chebyshev interpolation Platte & Trefethen (2008) ... "combine the **feel of symbolics** with the **speed of numerics**."



Based on analyticity conditions we have derived widely applicable conditions that imply

– (sub)exponential convergence of $I_N(Price^{K,T,q}) \to Price^{K,T,q}$,

– convergence results for the derivatives.

Conditions

Let the parameter set $\mathcal{P} = \mathcal{P}^1 \times \mathcal{P}^2 \subset \mathbb{R}^D$ be a **hyperrectangle**.
Let $\varrho \in (1,\infty)^D$ with $\varrho^1 := (\varrho_1, \dots, \varrho_m)$ and $\varrho^2 := (\varrho_{m+1}, \dots, \varrho_D)$ and let weight $\eta \in \mathbb{R}^d$.

(A1) **Integrability** of $x \mapsto e^{\langle \eta, x \rangle} f^K(x)$ for all $K \in \mathcal{P}^1$.

(A2) **Analyticity** of $K \mapsto \widehat{f^K}(z - i\eta)$ in $B(\mathcal{P}^1, \varrho^1)$ for all $z \in \mathbb{R}^d$.

(A3) **Exponential moment** $E\big(e^{-\langle \eta, X_T^q \rangle}\big) < \infty$ for all $(T,q) \in \mathcal{P}^2$.

(A4) **Analyticity** of $(T,q) \mapsto \varphi^{T,q}(z + i\eta)$ in $B(\mathcal{P}^2, \varrho^2)$ for all $z \in \mathbb{R}^d$.

(A5) **Uniform bound**: There exists $h \in L^1(\mathbb{R}^d)$ such that

$$\max_{(K,T,q) \in B(\mathcal{P},\varrho)} \left| \widehat{f^K}(-z - i\eta) \varphi^{T,q}(z + i\eta) \right| \le h(z) \quad \text{for all } z \in \mathbb{R}^d.$$

Convergence of Chebyshev Interpolation for POP

Theorem (Gaß, G., Mahlstedt, Mair (2016))

Let $\varrho \in (1,\infty)^D$ and weight $\eta \in \mathbb{R}^d$. Conditions (A1)–(A5) imply

$$\max_{(K,T,q)\in\mathcal{P}} \left| Price^{(K,T,q)} - I_{\overline{N}}(Price^{(\cdot)})(K,T,q) \right| \leq C \underline{\varrho}^{-\underline{N}},$$

where $\underline{\varrho} = \min_{1\leq i\leq D} \varrho_i$ and $\underline{N} = \min_{1\leq i\leq D} N_i$.

Univariate case: Bernstein (1912), Trefethen (2013),
proof of general result: Sauter and Schwab (2004),
improved error bounds: G. and Mahlstedt (2016)

Converge Analysis for Derivatives

Corollary (Gaß, G., Mahlstedt, Mair (2016))

Conditions (A1)–(A5) imply for every $m \in \mathbb{N}_0$ there exists a constant $C(m) > 0$ such that

$$\|Price^{(\cdot)} - I_{\overline{N}}(Price^{(\cdot)})(\cdot)\|_{C^l(\mathcal{P})} \leq C(M)N^{-m}\|Price^{(\cdot)}\|_{C^{2(l+1)+D+m}(\mathcal{P})}$$

with

$$\|u\|_{C^l(\mathcal{P})} = \max_{|\alpha|\leq l} \max_{p\in\mathcal{P}} |\partial^\alpha u(p)|.$$

Compare Canuto and Quarteroni (1982).

Interaction of Approximation Errors at Nodal Points and Interpolation Errors

Corollary (Gaß, G., Mahlstedt, Mair (2016))

Let $\mathcal{P} \ni p \mapsto Price^p$ be given as before and assume that $\varepsilon^{p^{(k_1,\ldots,k_D)}} \leq \overline{\varepsilon}$ for all Chebyshev nodes $p^{(k_1,\ldots,k_D)}$. Then

$$\max_{p\in\mathcal{P}} \left| Price^p - I_{\overline{N}}(Price_{\varepsilon}^{(\cdot)})(p) \right| \leq C \underline{\varrho}^{-\underline{N}} + 2^D \overline{\varepsilon} \prod_{i=1}^{D} (N_i + 1).$$

Numerical Results

Numerical Experiments: European Call

Models:

Black and Scholes, Merton jump diffusion, a pure jump model, CGMY

- Free Parameters $K \in [1, 10]$ and $T \in [0.5, 2]$

Heston's stochastic volatility model

- Free Parameters $K \in [1, 10]$ and $v_0 \in [0.1^2, 0.4^2]$

Product: Call option on one asset.

Interpolation: In the 2 free parameters.

Numerical Experiments: European Call

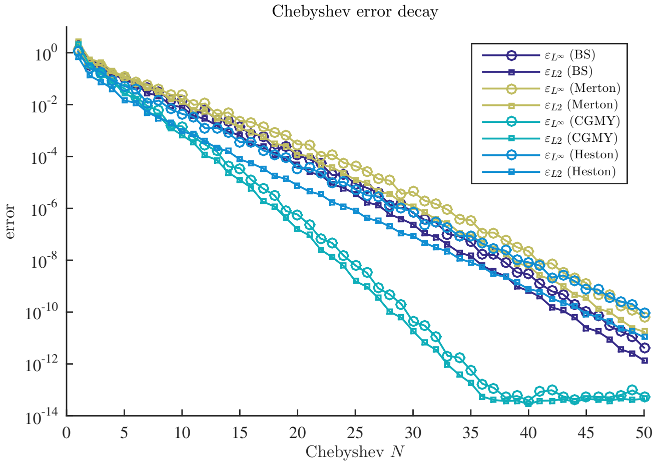


Figure : Convergence study for the BS, Merton, CGMY, Heston model

Numerical Experiments: Lookback Basket

Lookback basket option for d underlyings, $\overline{S}_j(T) := \max_{t \leq T} S_j(t)$,

$$f^K\big(S(t)_{0 \leq t \leq T}\big) = \left(\left(\frac{1}{d}\sum_{j=1}^{d} \overline{S}_j(T)\right) - K\right)^+.$$

This is a path-dependent and multi-asset option.

Computation – in virtually all models – requires Monte Carlo simulation.

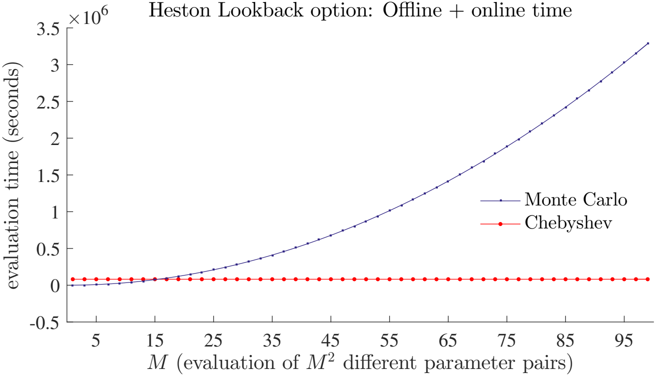
Model: Heston's stochastic volatility model, five underlyings, $d = 5$.

Interpolation: In 2 free parameters.

Efficiency: Compared to Monte Carlo

Lookback basket option on five assets in Heston's model.
The plot shows run-times for the computation of prices for $M^2 = 1, \dots, M^2 = 100^2$ different parameter values pairs

$$(\sigma, \varrho) \in [0.1, 0.5] \times [-1, 1]$$



Run-time comparison for same range of accuracy.

Total Run-times for $M = 100$

offline:

- 23 hours

online:

- 7 seconds
- 49 summands, each

Monte Carlo, by comparison, needs 39 days.

Example and outlook

Chebfun-3d Example: Spread option pricing in the bivariate Black-Scholes model, available on `http://www.chebfun.org/examples/applics/BlackScholes2D.html`

You can download the example and let it run. Here I let it run with slightly different specifications, namely the tolerance of chebfun3 is chosen 10^{-6}, whereas online it is set to 10^{-5} (so that the illustrative code is faster)

Low-rank 3d-tensor approximation of spread option prices in ϱ, K, T:

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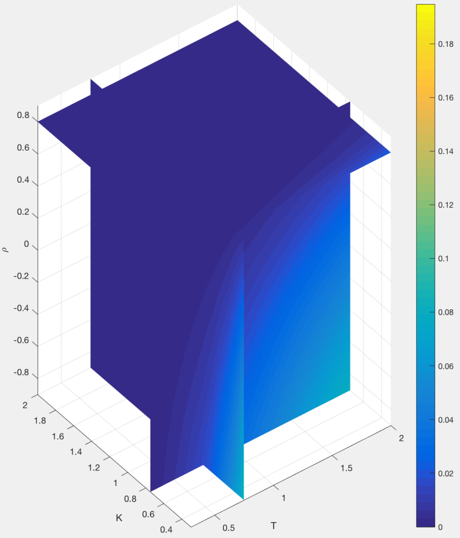
mmand Window

```
BlackScholes2D_SpreadOption_rho_T_K_v2

chebPrice =

   chebfun3 object
   cols: Inf x 6  chebfun
   rows: Inf x 16 chebfun
  tubes: Inf x 9  chebfun
   core: 6 x 16 x 9
 length: 12, 65, 17
 domain: [0.3, 2] x [0.3, 2] x [-0.9, 0.9]
 vertical scale = 0.19

Elapsed time is 226.981216 seconds.

time_price =

    2.0191


time_ChebPrice =

    0.0035


err =

   2.5630e-04

>>
```

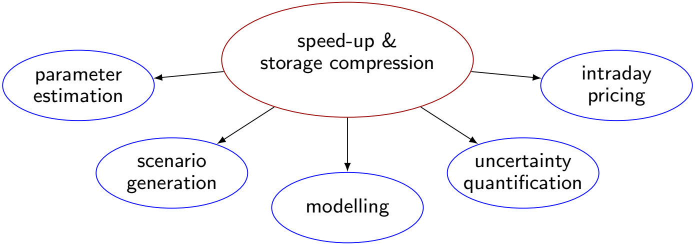
Figures – Figure 1

Figure 1



Outlook

- Monte Carlo + Chebyshev for **calibration** of rough vol model (with Christian Bayer WIAS Berlin, Jim Gatheral City Univ. NY)
- Approximation of the **implied volatility surface** (with Dilip Madan Univ. Maryland, Paul Herold, Christian Pötz, TU Munich)
- **More free parameters** (5+): low-rank tensor approximation technique (with Daniel Kressner, Francesco Statti, EPF Lausanne)
- **Applications:**



Paper:
Chebyshev Interpolation for Parametric Option Pricing, M. Gaß, K. Glau, M. Mahlstedt and M. Mair (2016), http://arxiv.org/abs/1505.04648

Literature

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Parametric Integration by Magic Point Empirical Interpolation, M. Gaß and K. Glau (2015), arxiv: 1511.08510