At-the-money short-term asymptotics under stochastic volatility models

Masaaki Fukasawa

Osaka University

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Joint work with O. El Euch, J. Gatheral and M. Rosenbaum

Overview

The aim of study:

- To derive an asymptotic expansion of the implied volatility.
- To prove the validity of the expansion (estimate the error).
- Examine if a model is consistent to empirical facts.
- The framework should include rough volatility models.
- Make calibration more efficient.

The plan of talk:

- On asymptotic methods for the implied volatility.
- At-the-money, short-term asymptotic expansions.
- Asymptotic skew and curvature.
- The SABR and rough Bergomi models.

Stochastic volatility models

$(\Omega, \mathcal{F}, P)$: a probability space with a filtration $\{\mathcal{F}_t; t \in \mathbb{R}\}$.
A log price process Z is assumed to follow (under Q)

$$\mathrm{d}Z_t = r\mathrm{d}t - \frac{1}{2}v_t\mathrm{d}t + \sqrt{v}_t\mathrm{d}B_t.$$

- $r \in \mathbb{R}$ stands for an interest rate,
- v is a progressively measurable positive process with respect to a smaller filtration $\{\mathcal{G}_t; t \in \mathbb{R}\}$, $\mathcal{G}_t \subset \mathcal{F}_t$.
- The $\{\mathcal{F}_t\}$-Brownian motion B is decomposed as

$$\mathrm{d}B_t = \rho_t\mathrm{d}W_t + \sqrt{1-\rho_t^2}\mathrm{d}W'_t,$$

  where W is an $\{\mathcal{G}_t\}$-BM and W' is independent of $\{\mathcal{G}_t\}$.
- ρ is a progressively measurable processes with respect to $\{\mathcal{G}_t\}$ and taking values in $(-1, 1)$.

A typical situation for stochastic volatility models is that (W, W') is a two dimensional $\{\mathcal{F}_t\}$-Brownian motion and $\{\mathcal{G}_t\}$ is the filtration generated by W, that is,

$$\mathcal{G}_t = \mathcal{N} \vee \sigma(W_s - W_r; r \leq s \leq t),$$

where $\mathcal{N}$ is the null sets of $\mathcal{F}$.

Example: $\rho \in (-1, 1)$ is constant and

$$v_t = \exp(Y_t), \ \ \mathrm{d}Y_t = \mu \mathrm{d}t + \eta \mathrm{d}W^H_t,$$

where W^H is a fractional Brownian motion driven by W as

$$W^H_t = c_H \int_{-\infty}^t (t-s)^{H-1/2} - (-s)_+^{H-1/2} \mathrm{d}W_t.$$

When $H = 1/2$, then the model is the log-normal SABR.

Of course, $\{\mathcal{G}_t\}$ can support a higher dimensional BM or Lévy.

The Black-Scholes implied volatility

An arbitrage-free price $p(K, \theta)$ of a put option at time 0 with strike $K > 0$ and maturity $\theta > 0$ is given by

$$p(K, \theta) = e^{-r\theta} E^Q[(K - \exp(Z_\theta))_+ | \mathcal{F}_0]$$
$$= e^{-r\theta} \int_0^K Q(\log x \geq Z_\theta | \mathcal{F}_0) \mathrm{d}x.$$

Denote by $p_{\mathrm{BS}}(K, \theta, \sigma)$ the put option price with strike price K and maturity θ under the Black-Scholes model with volatility parameter $\sigma > 0$.

Given a put option price $p(K, \theta)$, $K = Fe^k$, $F = e^{r\theta}$, the implied volatility $\sigma_{\mathrm{BS}}(k, \theta)$ is defined through

$$p_{\mathrm{BS}}(K, \theta, \sigma_{\mathrm{BS}}(k, \theta)) = p(K, \theta).$$

Asymptotic analyses for stochastic volatility

1. Perturbation expansions:
   - Introduce an artificial parameter ϵ as $v_t = v_t^\epsilon$.
   - Consider
   $$\int_0^\theta v_t^\epsilon \mathrm{d}t \to \int_0^\theta v_t^0 \mathrm{d}t =: \sigma^2\theta$$
   which is deterministic.
   - The limit model ($\epsilon \to 0$) is the Black-Scholes.
   - regular or singular: Yoshida's martingale expansion works.
   a) small vol-of-vol (Lewis, Bergomi-Guyon)
   b) multi-scale (Fouque-Papanicolaou-Sircar-Solna)
2. Short-term (small time-to-maturity) asymptotics:
   - $\theta \to 0$.
   - The limit is degenerate:
   $$Z_\theta = r\theta - \frac{1}{2}\int_0^\theta v_t \mathrm{d}t + \int_0^\theta \sqrt{v_t}\mathrm{d}B_t = O(\theta) + O(\sqrt{\theta}).$$
   - After rescaling as $\theta^{-1/2}Z_\theta$, the limit model is the Bachelier.

Short-term asymptotics

1. Large deviation
   - Does not rescale Z^θ.
   - For the transition density $p_\theta(x,y)$ of Z^θ,

$$-2\theta \log p_\theta(x,y) \to \inf\left\{\|h\|_2^2; \varphi(0) = x, \varphi(1) = y, \dot{\varphi} = vh\right\}$$

   - Heat kernel expansion, the SABR formula, ...
2. Edgeworth expansion
   - Rescale $\theta^{-1/2} Z_\theta$ to get a normal limit law.
   - Yoshida, Kunitomo-Takahashi (small diffusion expansions)
   - Medvedev-Scaillet: rescale as $z = \dfrac{k}{\sigma_{\mathrm{BS}}(k,\theta)\sqrt{\theta}}$ to expand $\sigma_{\mathrm{BS}}(k,\theta)$.
   - Have a closer look around at-the-money $k = 0$.
   - Here, we give a rigorous approach under a mild condition (in particular, we do not rely on the Malliavin calculus).

The strategy

Stochastic expansion

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Characteristic function expansion

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Density expansion

$\Downarrow$

Put option price expansion

$\Downarrow$

Implied volatility expansion

$\Downarrow$

Asymptotic skew and curvature

Each parts are in fact not very new...

Watanabe, Yoshida, Kunitomo and Takahashi

Rem. a different approach in F. (2017), only for the 1st order.

The framework

Recall

$$\mathrm{d}Z_t = r\mathrm{d}t - \frac{1}{2}v_t\mathrm{d}t + \sqrt{v}_t\mathrm{d}B_t, \ \ \mathrm{d}B_t = \rho_t\mathrm{d}W_t + \sqrt{1-\rho_t^2}\mathrm{d}W_t'.$$

Denote by E_0 and $\|\cdot\|_p$ respectively the expectation and the L^p norm under the regular conditional probability measure given $\mathcal{F}_0$, of which the existence is assumed.

Define the forward variance curve $v_0(t)$ by

$$v_0(t) = E_0[v_t] = E^Q[v_t|\mathcal{F}_0].$$

We impose the following technical condition:

$$\sup_{\theta\in(0,1)}\left\|\frac{1}{\theta}\int_0^\theta v_t\mathrm{d}t\right\|_p < \infty, \ \ \sup_{\theta\in(0,1)}\left\|\left\{\frac{1}{\theta}\int_0^\theta v_t(1-\rho_t^2)\mathrm{d}t\right\}^{-1}\right\|_p < \infty. \tag{1}$$

Stochastic expansion

Let

$$M_\theta = \int_0^\theta \sqrt{v_t}\mathrm{d}B_t, \;\; \langle M\rangle_\theta = \int_0^\theta v_t \mathrm{d}t, \;\; \sigma_0(\theta) = \sqrt{\int_0^\theta v_0(t)\mathrm{d}u}.$$

We assume the following asymptotic structure: there exists a family of random vectors

$$\left\{(M_\theta^{(0)}, M_\theta^{(1)}, M_\theta^{(2)}, M_\theta^{(3)}); \theta \in (0,1)\right\}, \quad \sup_{\theta\in(0,1)} \|M_\theta^{(i)}\|_p < \infty$$

for all $p > 0$ and $i = 1, 2, 3$ such that $\exists H > 0$ and $\exists \epsilon > 0$,

$$\begin{aligned} &\lim_{\theta\to 0} \theta^{-2H-2\epsilon} \left\| \frac{M_\theta}{\sigma_0(\theta)} - M_\theta^{(0)} - \theta^H M_\theta^{(1)} - \theta^{2H} M_\theta^{(2)} \right\|_{1+\epsilon} = 0, \\ &\lim_{\theta\to 0} \theta^{-H-2\epsilon} \left\| \frac{\langle M\rangle_\theta}{\sigma_0(\theta)^2} - 1 - \theta^H M_\theta^{(3)} \right\|_{1+\epsilon} = 0. \end{aligned} \tag{2}$$

Good to remember $E_0[M_\theta^2] = E_0[\langle M\rangle_\theta] = \sigma_0(\theta)^2 = O(\theta)$.

Further, we assume that the law of $M_\theta^{(0)}$ is standard normal for all $\theta > 0$ and the derivatives

$$\begin{aligned} a_\theta^{(i)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{E_0[M_\theta^{(i)}|M_\theta^{(0)} = x]\phi(x)\right\}, \quad i = 1, 2, 3, \\ b_\theta(x) &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{E_0[|M_\theta^{(1)}|^2|M_\theta^{(0)} = x]\phi(x)\right\} \end{aligned} \tag{3}$$

exist in the Schwartz space, where ϕ is the std. normal density.

Example: if $\mathrm{d}\sqrt{v_t} = c(\sqrt{v_t})\mathrm{d}W_t$, then by the Itô-Taylor,

$$\begin{aligned} M_\theta &\approx \int_0^\theta (\sqrt{v_0} + c(\sqrt{v_0})W_t) + c'(\sqrt{v_0})\int_0^t W_s \mathrm{d}W_s)\mathrm{d}B_t \\ &= \sqrt{v_0}\theta^{1/2} \\ &\times \left\{\hat{B}_1 + \frac{c(\sqrt{v_0})}{\sqrt{v_0}}\theta^{1/2}\int_0^1 \hat{W}_t \mathrm{d}\hat{B}_t + \frac{c'(\sqrt{v_0})}{\sqrt{v_0}}\theta\int_0^1\int_0^t \hat{W}_s \mathrm{d}\hat{W}_s \mathrm{d}\hat{B}_t\right\}. \end{aligned}$$

Characteristic function expansion I

Let $X_\theta = (Z_\theta - Z_0 - r\theta)/\sigma_0(\theta)$ and

$$Y_\theta = M_\theta^{(0)} + \theta^H M_\theta^{(1)} + \theta^{2H} M_\theta^{(2)} - \frac{\sigma_0(\theta)}{2}\left(1 + \theta^H M_\theta^{(3)}\right).$$

Lemma: Under (2), for any $\alpha \in \mathbb{N} \cup \{0\}$,

$$\sup_{|u| \le \theta^{-\epsilon}} |E_0[X_\theta^\alpha e^{iuX_\theta}] - E_0[Y_\theta^\alpha e^{iuY_\theta}]| = o(\theta^{2H+\epsilon}).$$

Proof: Since $|e^{ix} - 1| \le |x|$, we have

$$\begin{aligned}|E_0[X_\theta^\alpha e^{iuX_\theta}] - E_0[Y_\theta^\alpha e^{iuY_\theta}]| &\le E_0[|X_\theta^\alpha - Y_\theta^\alpha|] + uE_0[|Y_\theta|^\alpha |X_\theta - Y_\theta|] \\ &\le C(\alpha,\epsilon)(1 + |u|)\|X_\theta - Y_\theta\|_{1+\epsilon}\end{aligned}$$

for some constant $C(\alpha,\epsilon) > 0$. Since $\sigma_0(\theta) = O(\theta^{1/2})$, we obtain the result. ////

Characteristic function expansion II

Lemma: For any $\delta \in [0,(H-\epsilon)/3)$ and any $\alpha \in \mathbb{N} \cup \{0\}$,

$$\sup_{|u|\le\theta^{-\delta}} \Bigg| E_0[Y_\theta^\alpha e^{iuY_\theta}] - E_0\left[e^{iuM_\theta^{(0)}} \left((M_\theta^{(0)})^\alpha + A(\alpha,u,M_\theta^{(0)}) + B(\alpha,u,M_\theta^{(0)}) \right) \right] \Bigg| = o(\theta^{2H+\epsilon}),$$

where

$$A_\theta(\alpha,u,x) = \left(iux^\alpha + \alpha x^{\alpha-1} \right) (E_0[Y_\theta | M_\theta^{(0)} = x] - x),$$

$$B_\theta(\alpha,u,x) = \left(-\frac{u^2}{2} x^\alpha + iux^{\alpha-1} + \frac{\alpha(\alpha-1)}{2} x^{\alpha-2} \right) E_0[|M_\theta^{(1)}|^2 | M_\theta^{(0)} = x].$$

Proof: This follows from the fact that

$$\left| e^{ix} - 1 - ix + \frac{x^2}{2} \right| \le \frac{|x|^3}{6}$$

for all $x \in \mathbb{R}$. ////

Characteristic function expansion III

Lemma: Define $q_\theta(x)$ by

$$q_\theta(x) = \phi(x) - \theta^H a_\theta^{(1)}(x) - \theta^{2H} a_\theta^{(2)}(x) - \frac{\sigma_0(\theta)}{2}(x\phi(x) - \theta^H a_\theta^{(3)}(x)) + \frac{\theta^{2H}}{2} b_\theta(x), \tag{4}$$

where $a_\theta^{(i)}$ and b_θ are defined by (3). Then,

$$\int_{\mathbb{R}} e^{iux} x^\alpha q_\theta(x) \mathrm{d}x = E_0\left[e^{iuM_\theta^{(0)}}\left((M_\theta^{(0)})^\alpha + A(\alpha, u, M_\theta^{(0)}) + B(\alpha, u, M_\theta^{(0)})\right)\right].$$

Proof: This follows from integration by parts. ////

Density expansion I

Lemma: For any $\alpha, j \in \mathbb{N} \cup \{0\}$,

$$\sup_{\theta \in (0,1)} \int |u|^j |E_0[X_\theta^\alpha e^{iuX_\theta}]| \mathrm{d}u < \infty$$

Proof: Since the distribution of X_θ is Gaussian conditionally on $\mathcal{G}_\theta$, it admits a density $p_\theta(x)$ under $Q(\cdot|\mathcal{F}_0)$. Furthermore, the density function is in the Schwartz space $\mathcal{S}$ by (1). Therefore,

$$\begin{aligned}\int |u|^j |E_0[X_\theta^\alpha e^{iuX_\theta}]| \mathrm{d}u &= \int \left| \int u^j x^\alpha e^{iux} p_\theta(x) \mathrm{d}x \right| \mathrm{d}u \\ &= \int \left| \int e^{iux} \partial_x^j (x^\alpha p_\theta(x)) \mathrm{d}x \right| \mathrm{d}u < \infty\end{aligned}$$

since the Fourier transform is a map from $\mathcal{S}$ to $\mathcal{S}$. ////

Density expansion II

Theorem: The law of X_θ admits a density p_θ and for any $\alpha \in \mathbb{N} \cup \{0\}$,

$$\sup_{x \in \mathbb{R}} (1 + x^2)^\alpha |p_\theta(x) - q_\theta(x)| = o(\theta^{2H}) \tag{5}$$

as $\theta \to 0$, where q_θ is defined by (4).
Proof: As seen in the proof of Lemma, the density p_θ exists in the Schwartz space. By the Fourier identity

$$\begin{aligned}
&(1 + x^2)^\alpha |p_\theta(x) - q_\theta(x)| \\
&= \frac{1}{2\pi} \left| \int \int e^{iuy} (1 + y^2)^\alpha (p_\theta(y) - q_\theta(y)) \mathrm{d}y e^{-iux} \mathrm{d}u \right| \\
&= \frac{1}{2\pi} \left\{ \int_{|u| \le \theta^{-\delta}} |\cdot| \mathrm{d}u + \int_{|u| \ge \theta^{-\delta}} |\cdot| \mathrm{d}u \right\}.
\end{aligned}$$

Combining the lemmas in the previous section, taking $\delta \in (0, \min\{\epsilon, (H-\epsilon)/3\})$, we have

$$\int_{|u|\leq\theta^{-\delta}} \left| \int e^{iuy}(1+y^2)^{\alpha}(p_\theta(y) - q_\theta(y))\mathrm{d}y \right| \mathrm{d}u = o(\theta^{2H}).$$

On the other hand,

$$\begin{aligned} &\int_{|u|\geq\theta^{-\delta}} \left| \int e^{iuy}(1+y^2)^{\alpha} p_\theta(y)\mathrm{d}y \right| \mathrm{d}u \\ &\leq \theta^{j\delta} \int_{|u|\geq\theta^{-\delta}} |u|^j |E_0[(1+X_\theta^2)^{\alpha} e^{iuX_\theta}]| \mathrm{d}u = O(\theta^{j\delta}) \end{aligned}$$

for any $j \in \mathbb{N}$ by Lemma. The remainder

$$\int_{|u|\geq\theta^{-\delta}} \left| \int e^{iuy}(1+y^2)^{\alpha} q_\theta(y)\mathrm{d}y \right| \mathrm{d}u$$

is handled in the same manner. ////

Put option price expansion I

Denoting by p_θ the density of X_θ as before,

$$\frac{p(Fe^{\sigma_0(\theta)z},\theta)}{F\sigma_0(\theta)} = e^{-r\theta}\int_{-\infty}^{z}\int_{-\infty}^{\zeta} p_\theta(x)\mathrm{d}x e^{\sigma_0(\theta)\zeta}\mathrm{d}\zeta.$$

Lemma: Let $q_\theta(x)$, $\theta > 0$ be a family of functions on $\mathbb{R}$. If

$$\sup_{x\in\mathbb{R}}(1+x^2)^\alpha|p_\theta(x) - q_\theta(x)| = o(\theta^\beta)$$

for some $\alpha > 5/4$ and $\beta > 0$, then for any $z_0 \in \mathbb{R}$,

$$\frac{p(Fe^{\sigma_0(\theta)z},\theta)}{F\sigma_0(\theta)} = e^{-r\theta}\int_{-\infty}^{z}\int_{-\infty}^{\zeta} q_\theta(x)\mathrm{d}x e^{\sigma_0(\theta)\zeta}\mathrm{d}\zeta + o(\theta^\beta)$$

uniformly in $z \leq z_0$.

Put option price expansion II

Proof: By the Cauchy-Schwarz inequality,

$$
\begin{aligned}
& e^{-r\theta} \int_{-\infty}^{z} \int_{-\infty}^{\zeta} |p_\theta(x) - q_\theta(x)| \mathrm{d}z e^{\sigma_0(\theta)\zeta} \mathrm{d}\zeta \\
& \leq e^{-r\theta} \int_{-\infty}^{z} \sqrt{\int_{-\infty}^{\zeta} \frac{\mathrm{d}x}{(1+x^2)^{2\alpha-1}}} \\
& \times \sqrt{\int_{-\infty}^{\zeta} (1+x^2)^{2\alpha-1} |p_\theta(x) - q_\theta(x)|^2 \mathrm{d}z e^{\sigma_0(\theta)\zeta} \mathrm{d}\zeta} \\
& \leq \pi e^{-r\theta+\sigma_0(\theta)z} \sup_{x\in\mathbb{R}} (1+x^2)^{\alpha} |p_\theta(x) - q_\theta(x)| \\
& \times \int_{-\infty}^{z} \sqrt{\int_{-\infty}^{\zeta} \frac{\mathrm{d}x}{(1+x^2)^{2\alpha-1}}} \mathrm{d}\zeta,
\end{aligned}
$$

which is $o(\theta^\beta)$ if $\alpha > 5/4$. ////

Put option price expansion III

Theorem: Suppose we have (5) with q_θ of the form

$$q_\theta(x) = \phi(x)\left\{1 - \frac{\sigma_0(\theta)}{2}H_1(x) + \kappa_3(\theta)(H_3(x) - \sigma_0(\theta)H_2(x))\theta^H + \left(\kappa_4 H_4(x) + \frac{\kappa_3(\theta)^2}{2}H_6(x)\right)\theta^{2H}\right\},$$

where H_k is the kth Hermite polynomial :

$$H_1(x) = x,\ H_2(x) = x^2-1,\ H_3(x) = x^3-3x,\ H_4(x) = x^4-6x^2+3, \ldots$$

Then, for any $z_0 \in \mathbb{R}$,

$$\frac{p(Fe^{\sigma_0(\theta)z}, \theta)}{Fe^{-r\theta}\sigma_0(\theta)} = \frac{\Phi(z)e^{\sigma_0(\theta)z} - \Phi(z - \sigma_0(\theta))}{\sigma_0(\theta)} + \phi(z)\left\{\kappa_3 H_1(z)e^{\sigma_0(\theta)z}\theta^H + \left(\kappa_4 H_2(z) + \frac{\kappa_3^2}{2}H_4(z)\right)\theta^{2H}\right\} + o(\theta^{2H})$$

uniformly in $z \le z_0$, where $\kappa_3 = \kappa_3(\theta)$.

Implied volatility expansion

Theorem: Under the same condition as before,

$$\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta) = \kappa_2\left\{1+\frac{\kappa_3}{\kappa_2}z\theta^H+\left(\frac{3\kappa_3^2}{2}-\kappa_4+\frac{\kappa_4-3\kappa_3^2}{\kappa_2^2}z^2\right)\theta^{2H}\right\} + o(\theta^{2H})$$

when $H < 1/2$ and

$$= \kappa_2\left\{1+\frac{\kappa_3}{\kappa_2}z\sqrt{\theta}+\left(\frac{3\kappa_3^2}{2}-\kappa_4+\left(\frac{\kappa_4-3\kappa_3^2}{\kappa_2^2}+\frac{\kappa_3}{2\kappa_2}\right)z^2\right)\theta\right\}+o(\theta)$$

when $H = 1/2$, where $\kappa_2 = \kappa_2(\theta) = \sigma_0(\theta)/\sqrt{\theta}$ and $\kappa_3 = \kappa_3(\theta)$.

Note that $\kappa_2(\theta) = \sqrt{\text{the averaged forward variance}}$.

Asymptotics for at-the-money skew and curvature

Theorem: Under the same condition as before,

$$\partial_k \sigma_{\mathrm{BS}}(0,\theta) = \kappa_3(\theta)\theta^{H-1/2} + o(\theta^{2H-1/2}),$$

$$\partial_k^2 \sigma_{\mathrm{BS}}(0,\theta) = 2\frac{\kappa_4 - 3\kappa_3(\theta)^2}{\kappa_2(\theta)}\theta^{2H-1} + \kappa_3(\theta)\theta^{H-1/2} + o(\theta^{2H-1}).$$

Proof: Combine the previous expansions and

$$\partial_k \sigma_{\mathrm{BS}}(k,\theta) = \frac{Q(k \geq \sigma_0(\theta)X_\theta|\mathcal{F}_0) - \Phi(f_+(k,\theta))}{\sqrt{\theta}\phi(f_+(k,\theta))},$$

$$\partial_k^2 \sigma_{\mathrm{BS}}(k,\theta) = \frac{p_\theta(k/\sigma_0(\theta))}{\sigma_0(\theta)\sqrt{\theta}\phi(f_+(k,\theta))} - \sigma_{\mathrm{BS}}(k,\theta)\partial_k f_-(k,\theta)\partial_k f_+(k,\theta),$$

where

$$f_\pm(k,\theta) = \frac{k}{\sqrt{\theta}\sigma_{\mathrm{BS}}(k,\theta)} \pm \frac{\sqrt{\theta}\sigma_{\mathrm{BS}}(k,\theta)}{2}.$$

The rough Bergomi model

Let $\rho_t = \rho \in (-1, 1)$ be a constant and

$$\mathrm{d} \log v_t = \eta \mathrm{d} W_t^H + \text{deterministic drift},$$

where $\eta > 0$ is a constant and W^H is a fractional Brownian motion with the Hurst parameter $H \in (0, 1/2)$, given as

$$W_t^H = c_H \int_{-\infty}^{t} (t-s)^{H-1/2} - (-s)_+^{H-1/2} \mathrm{d} W_s$$

with a normalizing constant $c_H > 0$. Since v_t is log-normally distributed, (1) holds by Jensen's inequality. We have

$$v_t = v_0(t) \exp \left\{ \eta_H \sqrt{2H} \int_0^t (t-s)^{H-1/2} \mathrm{d} W_s - \frac{\eta_H^2}{2} t^{2H} \right\},$$

where $\eta_H = \eta c_H / \sqrt{2H}$. Note that $v_0(t)$ is rough.

The Hermite polynomials

Let H_k, $k = 0, 1, \ldots$ be the Hermite polynomials:

$$H_k(x) = (-1)^k e^{x^2/2} \frac{\mathrm{d}^k}{\mathrm{d}x^k} e^{-x^2/2}$$

and $H_k(x, a) = a^{k/2} H_k(x/\sqrt{a})$ for $a > 0$. As is well-known, we have

$$\exp\left\{ux - \frac{au^2}{2}\right\} = \sum_{k=0}^{\infty} H_k(x, a) \frac{u^k}{k!}$$

and for any continuous local martingale M and $n \in \mathbb{N}$,

$$\mathrm{d}L_t^{(n)} = nL_t^{(n-1)} \mathrm{d}M_t, \tag{6}$$

where $L^{(k)} = H_k(M, \langle M \rangle)$ for $k \in \mathbb{N}$.

Time-change for the rBergomi model

Define $\hat{W}$, $\hat{W}'$, $\hat{B}$ by

$$\hat{W}_t = \frac{1}{\sigma_0(\theta)} \int_0^{\tau^{-1}(t)} \sqrt{v_0(s)}\mathrm{d}W_s, \quad \hat{W}'_t = \frac{1}{\sigma_0(\theta)} \int_0^{\tau^{-1}(t)} \sqrt{v_0(s)}\mathrm{d}W'_s$$

and $\hat{B} = \rho\hat{W} + \sqrt{1-\rho^2}\hat{W}'$, where

$$\tau(s) = \frac{1}{\sigma_0(\theta)^2} \int_0^s v_0(t)\mathrm{d}t.$$

Then, $(\hat{W}, \hat{W}')$ is a 2-dimensional Brownian motion under E_0 and for any square-integrable function f,

$$\int_0^a f(s)\mathrm{d}W_s = \sigma_0(\theta) \int_0^{\tau(a)} \frac{f(\tau^{-1}(t))}{\sqrt{v_0(\tau^{-1}(t))}}\mathrm{d}\hat{W}_t.$$

Therefore,

$$M_\theta = \sigma_0(\theta) \int_0^1 \exp\left\{\theta^H F_t^t - \frac{\eta_H^2}{4} |\tau^{-1}(t)|^{2H}\right\} \mathrm{d}\hat{B}_t$$

where

$$F_u^t = \eta_H \sqrt{\frac{H}{2}} \frac{\sigma_0(\theta)}{\theta^H} \int_0^u \frac{(\tau^{-1}(t) - \tau^{-1}(s))^{H-1/2}}{\sqrt{v_0(\tau^{-1}(s))}} \mathrm{d}\hat{W}_s, \quad u \in [0, t].$$

Let

$$G_t^{(k)} = H_k(F_t^t, \langle F^t \rangle_t).$$

Then, we have

$$M_\theta = \sigma_0(\theta) \int_0^1 \exp\left\{-\frac{\eta_H^2}{8} |\tau^{-1}(t)|^{2H}\right\} \exp\left\{\theta^H F_t^t - \frac{\theta^{2H}}{2} \langle F^t \rangle_t\right\} \mathrm{d}\hat{B}_t$$

$$= \sigma_0(\theta) \int_0^1 \exp\left\{-\frac{\eta_H^2}{8} |\tau^{-1}(t)|^{2H}\right\} \sum_{k=0}^\infty G_t^{(k)} \frac{\theta^{Hk}}{k!} \mathrm{d}\hat{B}_t.$$

Stochastic expansion for the rBergomi model

Lemma: We have (2) with

$$M_\theta^{(0)} = \hat{B}_1,$$

$$M_\theta^{(1)} = \int_0^1 h_\theta(t) G_t^{(1)} \mathrm{d}\hat{B}_t,$$

$$M_\theta^{(2)} = \int_0^1 \left\{ \frac{h_\theta(t) - 1}{\theta^{2H}} + h_\theta(t) \frac{G_t^{(2)}}{2} \right\} \mathrm{d}\hat{B}_t,$$

$$M_\theta^{(3)} = 2 \int_0^1 F_t^t \mathrm{d}t,$$

where

$$h_\theta(t) = \exp\left\{ -\frac{\eta_H^2}{8} |\tau^{-1}(t)|^{2H} \right\}.$$

Density expansion for the rBergomi model

Theorem: We have (5) with

$$q_\theta(x) = \phi(x)\Bigg\{1 - \frac{\sigma_0(\theta)}{2}H_1(x) + \kappa_3(\theta)(H_3(x) - \sigma_0(\theta)H_2(x))\theta^H + \left(\kappa_4 H_4(x) + \frac{\kappa_3(\theta)^2}{2}H_6(x)\right)\theta^{2H}\Bigg\},$$

where

$$\kappa_3(\theta) = \rho\eta_H\sqrt{\frac{H}{2}}\frac{1}{\theta^H\sigma_0(\theta)^3}\int_0^\theta\int_0^t (t-s)^{H-1/2}\sqrt{v_0(s)}\mathrm{d}s v_0(t)\mathrm{d}t,$$

$$\kappa_4 = \frac{(1+2\rho^2)\eta_H^2 H}{(2H+1)^2(2H+2)} + \frac{\rho^2\eta_H^2 H\beta(H+3/2, H+3/2)}{2(H+1/2)^2}.$$

Brownian bridge

Since $M_\theta^{(0)} = \hat{B}_1$, computing $E_0[M_\theta^{(i)} | M_\theta^{(0)} = x]$ reduces to compute expectations of iterated integrals of Brownian bridge.

Lemma :

$$\hat{E}\left[\int_0^1 \int_0^t f(s,t)\mathrm{d}\hat{B}_s \mathrm{d}t\right] = H_1(x) \int_0^1 \int_0^t f(s,t)\mathrm{d}s\mathrm{d}t,$$

$$\hat{E}\left[\int_0^1 \int_0^t f(s,t)\mathrm{d}\hat{B}_s \mathrm{d}\hat{B}_t\right] = H_2(x) \int_0^1 \int_0^t f(s,t)\mathrm{d}s\mathrm{d}t,$$

$$\hat{E}\left[\int_0^1 \left(\int_0^t f(s,t)\mathrm{d}\hat{B}_s\right)^2 \mathrm{d}\hat{B}_t\right] = H_3(x) \int_0^1 \left(\int_0^t f(s,t)\mathrm{d}s\right)^2 \mathrm{d}t + H_1(x) \int_0^1 \int_0^t f(s,t)^2 \mathrm{d}s\mathrm{d}t.$$

and...

$$\hat{E}\left[\int_0^1 \left(\int_s^1 f(s,t)\mathrm{d}\hat{B}_t\right)^2 \mathrm{d}s\right]$$

$$= H_2(x)\int_0^1 \left(\int_s^1 f(s,t)\mathrm{d}t\right)^2 \mathrm{d}s + \int_0^1 \int_s^1 f(s,t)^2 \mathrm{d}t\mathrm{d}s,$$

$$\hat{E}\left[\left(\int_0^1 \int_0^t f(s,t)\mathrm{d}\hat{B}_s\mathrm{d}\hat{B}_t\right)^2\right] = H_4(x)\left(\int_0^1 \int_0^t f(s,t)\mathrm{d}s\mathrm{d}t\right)^2$$

$$+ H_2(x)\int_0^1 \left(\int_0^t f(s,t)\mathrm{d}s + \int_t^1 f(t,u)\mathrm{d}u\right)^2 \mathrm{d}t$$

$$+ \int_0^1 \int_0^t f(s,t)^2 \mathrm{d}s\mathrm{d}t.$$

Concluding remarks

- There is no technical difficulty to go higher orders.
- The same approach works for the small vol-of-vol perturbation.
- The rBergomi model explains the power law of volatility skew (and curvature).
- When the forward variance curve is flat, an expansion of the Forde-Zhang rate function of large deviation gives the same expansion of the implied volatility. Cf. Bayer et al.
- When the forward variance curve is flat, the (formal) small vol-of-vol (Bergomi-Guyon) expansion given by Bayer et al. (2016) coincides with our expansion.