Hitting times of one dimensional diffusions and Monte Carlo approximation

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Outline of the presentation

1. Introduction
2. The forward method results
3. The backward method results
4. Some heuristics

One dimensional diffusion and exit time

In this talk, we consider a one dim. uniformly elliptic diffusion

$$X_t^x = x + \int_0^t b(X_s^x)ds + \int_0^t \sigma(X_s^x)dW_s,$$

and its exit time from the domain $D = (-\infty, L)$ with

$$\tau^x := \inf\{v \geqslant 0 : X_v^x \geqslant L\}, \quad \tau_t^x := \tau^x \wedge t.$$

We are interested in studying the collection of linear maps

$$P_t h(u,x) := \mathbb{E}\left[h(u+\tau_t^x, X_{\tau_t^x}^x)\right]$$

Two cases : Forward $\sigma \in C_b^2, b \in C_b^1$ and Backward $\sigma \in C_b^\alpha, b \in \mathcal{M}_b$.

- Motivations :
  - Financial mathematics : pricing and hedging of barrier options, ...
  - Default risk : pricing and hedging with a structural approach.

Related literature

- Numerical computation : $\mathbb{E}[h(X_T^x)\mathbb{I}_{\{\tau^x > T\}}]$ for smooth domain D
  - Elliptic diff. : Gobet (00), discrete & continuous Euler scheme
  - Hypo-elliptic diff. : Gobet&Menozzi (03)

$$\mathcal{E}_d(T,h,\Delta,x) := \mathbb{E}[h(\tilde{X}_T^{x,\Delta})\mathbb{I}_{\{\tau_d^{x,\Delta} > T\}}] - \mathbb{E}[h(X_T^x)\mathbb{I}_{\{\tau^x > T\}}] \asymp C\sqrt{\Delta}$$

$$\mathcal{E}_c(T,h,\Delta,x) := \mathbb{E}[h(\tilde{X}_T^{x,\Delta})\mathbb{I}_{\{\tau_c^{x,\Delta} > T\}}] - \mathbb{E}[h(X_T^x)\mathbb{I}_{\{\tau^x > T\}}] \approx C\Delta$$

  - discrete Euler scheme is easy to implement in any dimension.
  - continuous Euler scheme : dimension 1, more delicate in higher dimension.

Probabilistic representation & Monte Carlo simulation

One of our main results is a probabilistic representation of $(P_t)_{t\geqslant 0}$ in the same spirit as *Bally & Kohatsu-Higa* (AAP, 15)

$$\mathbb{E}[f(X_T^x)] = e^{\lambda T}\mathbb{E}[f(\bar{X}_T^\pi)\prod_{k=0}^{N_T-1}\lambda^{-1}\theta_{\zeta_{k+1}-\zeta_k}(\bar{X}^\pi_{\zeta_k},\bar{X}^\pi_{\zeta_{k+1}})]$$

Unbiased Monte Carlo path simulation requires :

- $\{N_t; t\geqslant 0\}$ Poisson process with parameter λ, Jump times : $\zeta_1 < \cdots < \zeta_n$
- Euler scheme with random times :

$$\bar{X}^\pi_{\zeta_{k+1}} = \bar{X}^\pi_{\zeta_k} + b(\bar{X}^\pi_{\zeta_k})(\zeta_{k+1}-\zeta_k) + \sigma(\bar{X}^\pi_{\zeta_k})(W_{\zeta_{k+1}} - W_{\zeta_k}).$$

$$\theta_t(x,y) := a''(y) + 2a'(y)H_1(a(x)t, y-x-b(x)t) + (a(y)-a(x))H_2(a(x)t, y-x-b(x)t) \\ - b'(y) - (b(y)-b(x))H_1(a(x)t, y-x-b(x)t).$$

$$H_i(c,x) := (g(c,x))^{-1}\partial_x^i g(c,x),\ i\geqslant 0.$$

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Theorem : Probabilistic representation

$\{N_t; t \geqslant 0\}$ Poisson process with parameter λ, Jump times : $\zeta_1 < \cdots < \zeta_{N(T)}$
Euler scheme : $\bar{X}^{\pi}_{\zeta_{j+1}} = \bar{X}^{\pi}_{\zeta_j} + \sigma(\bar{X}^{\pi}_{\zeta_j})(W_{\zeta_{j+1}} - W_{\zeta_j})$

$$\Gamma_N(x) = e^{\lambda T} \prod_{j=0}^{N-1} \lambda^{-1} \bar{\theta}_{\zeta_{j+1}-\zeta_j}(\bar{X}^{\pi}_{\zeta_j}, \bar{X}^{\pi}_{\zeta_{j+1}}) \in L^1(\Omega)$$

$$\bar{\Gamma}_N(x) = e^{\lambda T} \frac{(a(L) - a(\bar{X}^{\pi}_{\zeta_{N-1}}))}{a(\bar{X}^{\pi}_{\zeta_{N-1}})} \prod_{j=0}^{N-2} \lambda^{-1} \bar{\theta}_{\zeta_{j+1}-\zeta_j}(\bar{X}^{\pi}_{\zeta_j}, \bar{X}^{\pi}_{\zeta_{j+1}}) \in L^1(\Omega)$$

Then, for all $h \in \mathcal{M}_b$,

$$\mathbb{E}[h(\tau^x_T, X^x_{\tau^x_T})] = e^{\lambda T}\mathbb{E}[h((\zeta_{N(T)} + \bar{\tau}^{\zeta_{N(T)}, \bar{X}^{\pi}_{\zeta_{N(T)}}}) \wedge T, \bar{X}^{\pi}_{(\zeta_{N(T)} + \bar{\tau}^{\zeta_{N(T)}, \bar{X}^{\pi}_{\zeta_{N(T)}}}) \wedge T})\Gamma_{N(T)}(x)]$$

$$+ e^{\lambda T}\mathbb{E}[h(\zeta_{N(T)-1} + \bar{\tau}^{\zeta_{N(T)-1}, \bar{X}^{\pi}_{\zeta_{N(T)-1}}}, L)\mathbb{I}_{\{\bar{\tau}^{\zeta_{N(T)-1}, \bar{X}^{\pi}_{\zeta_{N(T)-1}}} \leqslant T - \zeta_{N(T)-1}\}} \bar{\Gamma}_{N(T)}(x)].$$

Weight definitions

Here, we define

$$\theta_t(x,z) = \mathbb{I}_{\{x<L\}} \left\{ \left(\frac{1}{2}a''(z) - b'(z)\right) + (a'(z) - b(z))\mu_t^1(x,z) + \frac{1}{2}(a(z) - a(x))\mu_t^2(x,z) \right\},$$

$$\bar{\theta}_t(x,z) = \theta_t(x,z)\Lambda_t(x,z)$$

$$\Lambda_t(x,z) := \mathbb{P}\left(\sup_{0\leqslant v\leqslant t} W_v \leqslant \frac{L-x}{\sigma(x)} | W_t = \frac{z-x}{\sigma(x)} \right) = \left\{ 1 - e^{-2\frac{(L-x)(L-x-(z-x))}{ta(x)}} \right\} \mathbb{I}_{\{x\leqslant L\}} \mathbb{I}_{\{z\leqslant L\}}$$

$$\mu_t^1(x,z) = H_1(a(x)t, z-x) - \frac{1}{a(x)t} \frac{2(L-x)}{(\exp(-\frac{2(z-L)(L-x)}{a(x)t}) - 1)},$$

$$\mu_t^2(x,z) = H_2(a(x)t, z-x) + \frac{1}{a^2(x)t^2} \frac{4(z-L)(L-x)}{(\exp(-\frac{2(z-L)(L-x)}{a(x)t}) - 1)}.$$

$$\bar{\tau}^x \sim f_{\bar{\tau}}(x,s) = \partial_s \mathbb{P}(\bar{\tau}^x \leqslant s) = \frac{L-x}{\sqrt{2\pi a(x)}s^{3/2}} \exp\left(-\frac{(L-x)^2}{2a(x)s}\right) \mathbb{I}_{\{x\leqslant L\}}.$$

Corollary : Integration by parts for killed process

Let $T > 0$. Then, for all $x \in (-\infty, L)$, one has

$$\mathbb{E}[h'(X_T^x)\mathbb{I}_{\{\tau^x > T\}}]$$
$$= -\mathbb{E}\left[h(\bar{X}_T^\pi)\Lambda_{T-\zeta_{N(T)}}(\bar{X}_{\zeta_{N(T)}}^\pi, \bar{X}_T^\pi)\mu^1_{T-\zeta_{N(T)}}(\bar{X}_{\zeta_{N(T)}}^\pi, \bar{X}_T^\pi)\Gamma_{N(T)}(x)\right].$$

For all $x \in (-\infty, L]$, the joint law of $(\tau_T^x, X_{\tau_T^x}^x)$ is given by the measure

$$p_T(x, dt, dz) := p^K(x,t)\delta_L(dz)dt + p_T^D(x,z)\delta_T(dt)dz$$

and, for some positive $C, c > 1$, for all $(t,z) \in (0,T] \times (-\infty, L]$:

$$p^K(x,t) \leqslant Ct^{-1/2}g(ct, L-x) \quad \text{and} \quad p_T^D(x,z) \leqslant Cg(cT, z-x),$$

$z \mapsto p_T^D(x,z) \in \mathcal{C}^{1+\alpha}((-\infty, L])$, $t \mapsto p^K(x,t) \in \mathcal{C}^{\alpha/2}((0,T])$, for all $\alpha < 1$.

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$$\vartheta_t(z,x) := \frac{1}{2}(a(x)-a(z))\hat{\mu}_t^2(z,x) + b(x)\hat{\mu}_t^1(z,x)$$

$$\hat{\mu}_t^1(x,z) := H_1(a(z)t, z-x) - \frac{1}{a(z)t}\frac{2(L-z)}{(\exp(\frac{2(L-z)(L-x)}{a(z)t})-1)}$$

$$\hat{\mu}_t^2(x,z) := H_2(a(z)t, z-x) - \frac{1}{a^2(z)t^2}\frac{4(L-z)(L-x)}{(\exp(\frac{2(L-z)(L-x)}{a(x)t})-1)}.$$

$$\bar{\vartheta}_t(z,x) := \vartheta_t(z,x)\Lambda_t(z,x)$$

$$\hat{\vartheta}_t^i(z,x) := \begin{cases} \frac{\bar{q}_t^z(x,z)}{\bar{q}_t^x(x,z)} & i=0, \\ \bar{\vartheta}_t(z,x)\frac{\bar{q}_t^z(x,z)}{\bar{q}_t^x(x,z)} & i=1,\dots,n-1. \end{cases}$$

Probabilistic representation in the Backward method

Let $T > 0$. Assume that $a \in \mathcal{C}^\alpha$ and $b \in \mathcal{M}_b$. Define

$$\bar{\Gamma}^D_N(z) = e^{\lambda T}\lambda^{-N}\prod_{i=0}^{N-1}\bar{\vartheta}_{\zeta_{i+1}-\zeta_i}(\bar{X}^{\pi,z}_{\zeta_i}, \bar{X}^{\pi,z}_{\zeta_{i+1}})$$

$$\bar{\Gamma}^K_N(x) := e^{\lambda T}\lambda^{-N}\mathbb{I}_{\{\zeta_N < t < T\}}\hat{\vartheta}_L(\bar{X}^{\pi,x}_{\zeta_N}, t-\zeta_N)\prod_{i=0}^{N-1}\hat{\vartheta}^i_{\zeta_{i+1}-\zeta_i}(\bar{X}^{\pi,x}_{\zeta_{i+1}}, \bar{X}^{\pi,x}_{\zeta_i})$$

Then, for all $(t,x) \in (0,T] \times (-\infty, L)$, the following probabilistic representation holds

$$p^D_T(x,z) = \mathbb{E}\Big[\bar{q}^{\bar{X}^{\pi,z}_{\zeta_N}}_{T-\zeta_N}(x, \bar{X}^{\pi,z}_{\zeta_N})\bar{\Gamma}^D_N(z)\Big],$$

$$p^K(x,t) = \mathbb{E}\Big[f^L_{\bar{T}}(\bar{X}^{\pi,x}_{\zeta_N}, t-\zeta_N)\bar{\Gamma}^K_N(x)\Big].$$

Corollary : Integration by parts formula

Let $T > 0$. Assume that $a \in \mathcal{C}^{\alpha}$ and $b \in \mathcal{M}_b$.

$$\partial_x p_T^D(x,z) = \mathbb{E}\left[\bar{q}_{T-\zeta_N}^{\bar{X}_{\zeta_N}^{\pi,z}}(x, \bar{X}_{\zeta_N}^{\pi,z})\hat{\mu}_{T-\zeta_N}^{1}(x, \bar{X}_{\zeta_N}^{\pi,z})\bar{\Gamma}_N^D(z)\right].$$

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Markov maps

$$\begin{aligned} P_t h(u,x) &= \mathbb{E}[h(u+\tau_t^x, X_{\tau_t^x}^x)] \\ &= h(u,x)\mathbb{I}_{\{x \geqslant L\}} \\ &\quad + \mathbb{I}_{\{x<L\}}\left\{\mathbb{E}[h(u+t, X_t^x)\mathbb{I}_{\{\tau^x>t\}}] + \mathbb{E}[h(u+\tau^x, L)\mathbb{I}_{\{\tau^x \leqslant t\}}]\right\}. \end{aligned}$$

Proxy semigroup with frozen diffusion coefficient : For $y \in \mathbb{R}$, consider

$$\bar{X}_t^x = x + \sigma(y) W_t, \quad \bar{\tau}^x = \inf\{t \geqslant 0 : \bar{X}_t^x \geqslant L\}, \ y \in \mathbb{R}.$$

and

$$\bar{P}_t^y h(u,x) = \mathbb{E}[h(u+\bar{\tau}_t^x, \bar{X}_{\bar{\tau}_t^x}^x)].$$

Infinitesimal generators

From Itô's rule, one obtains :

Proposition : Infinitesimal generators

For $h \in C_b^{1,2}(\mathbb{R}_+ \times]-\infty, L]) \cap \mathcal{M}_b(\mathbb{R}_+ \times \mathbb{R})$, one has

$$\begin{aligned}
\frac{(P_t h - h)(u,x)}{t} &\underset{t \downarrow 0}{\rightarrow} \mathcal{L}h(u,x) \\
&:= \mathbb{I}_{\{x<L\}} \left(b(x)\partial_2 h(u,x) + \frac{1}{2}a(x)\partial_2^2 h(u,x) + \partial_1 h(u,x) \right), \\
\frac{(\bar{P}_t^y h - h)(u,x)}{t} &\underset{t \downarrow 0}{\rightarrow} \bar{\mathcal{L}}^y h(u,x) \\
&:= \mathbb{I}_{\{x<L\}} \left(\frac{1}{2}a(y)\partial_2^2 h(u,x) + \partial_1 h(u,x) \right).
\end{aligned}$$

The proxy semigroup

$$\begin{aligned}
\bar{P}^y_t h(u,x) &= \mathbb{I}_{\{x \geqslant L\}} h(u,x) + \mathbb{I}_{\{x<L\}} \mathbb{E}[h(u+t, \bar{X}^x_t) \mathbb{I}_{\{\bar{\tau}^x > t\}}] + \mathbb{I}_{\{x<L\}} \mathbb{E}[h(u+\bar{\tau}^x, L) \mathbb{I}_{\{\bar{\tau}^x \leqslant t\}}] \\
&= \mathbb{I}_{\{x \geqslant L\}} h(u,x) + \mathbb{I}_{\{x<L\}} \int_{-\infty}^{L} h(u+t,z) \bar{q}^y_t(x,z) dz + \mathbb{I}_{\{x<L\}} \int_0^t h(u+s,L) f^y_{\bar{\tau}}(x,s) ds \\
&= \mathbb{I}_{\{x \geqslant L\}} h(u,x) + S^y_t h(u,x) + K^y_t h(u,x) \\
&= \mathbb{I}_{\{x \geqslant L\}} h(u,x) + \mathbb{I}_{\{x<L\}} \int_u^{u+t} \int_{-\infty}^{L} h(s,z) \left[\delta_{u+t}(ds) \bar{q}^y_t(x,z) dz + \delta_L(dz) f^y_{\bar{\tau}}(x, s-u) ds \right].
\end{aligned}$$

with (from the reflection principle)

$$\bar{q}^y_t(x,z) = \frac{\mathbb{P}(\bar{X}^x_t \in dz, \sup_{v \in [0,t]} \bar{X}^x_v < L)}{dz} = \left(g(a(y)t, z-x) - g(a(y)t, z+x-2L)\right) \mathbb{I}_{\{x,z \leqslant L\}},$$

$$f^y_{\bar{\tau}}(x,s) = \partial_s \mathbb{P}(\bar{\tau}^x \leqslant s) = \frac{L-x}{\sqrt{2\pi a(y)} s^{3/2}} \exp\left(-\frac{(L-x)^2}{2a(y)s} \right) \mathbb{I}_{\{x \leqslant L\}}.$$

Forward parametrix expansion

From now on, we freeze the coefficient to the starting point : $y = x$.

- Forward parametrix expansion consists in writing

$$P_T h(u,x) - \bar{P}_T h(u,x) = \int_0^T \partial_s(\bar{P}_{T-s}P_s)h(u,x)ds = \int_0^T \bar{P}_{T-s}(\mathcal{L} - \bar{\mathcal{L}})P_s h(u,x)ds$$

$$\overset{IBP}{=} \int_0^T ds\left(\bar{K}_{T-s}P_s h(u,x) + \bar{S}_{T-s}P_s h(u,x)\right)$$

with

$$\bar{K}_t h(u,x) = \bar{K}_t(x,L)h(u+t,L) \quad \text{and} \quad \bar{S}_t h(u,x) = \int_{-\infty}^{L} h(u+t,z)\bar{S}_t(x,z)dz$$

with

$$\bar{K}_t(x,L) = \mathbb{I}_{\{x<L\}}\frac{(a(L)-a(x))}{a(x)}f_{\bar{\tau}}^{x}(x,t)$$

$$\bar{S}_t(x,z) = \mathbb{I}_{\{x<L\}}\left\{\partial_z^2\left[\frac{1}{2}(a(z)-a(x))\bar{q}_t^x(x,z)\right] - \partial_z\left[b(z)\bar{q}_t^x(x,z)\right]\right\}$$

Expansion of the semigroup

Iterating the one step expansion, we get

$$
\begin{aligned}
P_T h(u,x) &= \bar{P}_T h(u,x) + \int_0^T ds_1 \left\{ \bar{K}_{T-s_1} h(u,x) + \bar{S}_{T-s_1} P_{s_1} h(u,x) \right\} \\
&= \bar{P}_T h(u,x) + \int_0^T ds_1 \left\{ \bar{K}_{T-s_1} h(u,x) + \bar{S}_{T-s_1} \bar{P}_{s_1} h(u,x) \right\} \\
&+ \int_0^T ds_1 \int_0^{s_1} ds_2 \left\{ \bar{K}_{T-s_1} \bar{K}_{s_1-s_2} P_{s_2} h(u,x) + \bar{K}_{T-s_1} \bar{S}_{s_1-s_2} P_{s_2} h(u,x) \right\} \\
&+ \int_0^T ds_1 \int_0^{s_1} ds_2 \left\{ \bar{S}_{T-s_1} \bar{K}_{s_1-s_2} P_{s_2} f(x,u) + \bar{S}_{T-s_1} \bar{S}_{s_1-s_2} P_{s_2} h(u,x) \right\} \\
&= \bar{P}_T h(u,x) + \int_0^T ds_1 \left\{ \bar{K}_{T-s_1} h(u,x) + \bar{S}_{T-s_1} \bar{P}_{s_1} h(u,x) \right\} \\
&+ \int_0^T ds_1 \int_0^{s_1} ds_2 \left\{ \bar{S}_{T-s_1} \bar{K}_{s_1-s_2} P_{s_2} f(x,u) + \bar{S}_{T-s_1} \bar{S}_{s_1-s_2} P_{s_2} h(u,x) \right\} \\
&= \bar{P}_T h(u,x) + \sum_{n=1}^{N} \int_{\Delta_n(T)} d\mathbf{s}_n \bar{S}_{T-s_1} \bar{S}_{s_1-s_2} \cdots \bar{S}_{s_{n-2}-s_{n-1}} \bar{K}_{s_{n-1}-s_n} \bar{P}_{s_n} h(u,x) \\
&+ \sum_{n=1}^{N} \int_{\Delta_n(T)} d\mathbf{s}_n \bar{S}_{T-s_1} \bar{S}_{s_1-s_2} \cdots \bar{S}_{s_{n-1}-s_n} \bar{P}_{s_n} h(u,x) + \mathcal{R}_T^N h(u,x)
\end{aligned}
$$

Defining

$$I^n_{s_0}h(u,x) = \begin{cases} \int_{\Delta_n(s_0)} d\mathbf{s}_n \left\{ \left(\prod_{i=0}^{n-1} \bar{S}_{s_i - s_{i+1}}\right) \bar{P}_{s_n} h(u,x) + \left(\prod_{i=0}^{n-2} \bar{S}_{s_i - s_{i+1}}\right) \bar{K}_{s_{n-1}-s_n} h(u,x) \right\}, n \geqslant 1, \\ \bar{P}_{s_0} h(u,x), \ n = 0, \end{cases}$$

one concludes

Theorem : Semigroup expansion

Let $T > 0$. For every $h \in C_b^{1,2}(\mathbb{R} + \times]-\infty, L]) \cap \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R})$ and for every $(t,u,x) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}$, one has

$$P_t h(u,x) = \sum_{n \geqslant 0} I_t^n h(u,x).$$

Then, the probabilistic interpretation follows...