# Irreducible paving map for martingale couplings in finite dimension

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# Outline

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# Existence of martingale couplings



- $\Omega = \mathbb{R}^d \times \mathbb{R}^d$ and $X$ and $Y$ are the two canonical random variables $\Omega \to \mathbb{R}^d$, $X : (x, y) \mapsto x$ and $Y : (x, y) \mapsto y$.
- The set of all martingale coupling probability laws between $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$:

$$\mathcal{M}(\mu,\nu) := \{\mathbb{P} \in \mathcal{P}(\Omega) \ : \quad \mathbb{P}\circ X^{-1} = \mu,\ \mathbb{P}\circ Y^{-1} = \nu \text{ and } \mathbb{E}^{\mathbb{P}}[Y|X] = X\}.$$

- By (Strassen 1964), we have the following equivalence

$$\mathcal{M}(\mu,\nu) \neq \emptyset \iff \mu \overset{convex}{\leq} \nu$$

i.e. $(\nu - \mu)[f] \geq 0$ for any $f$ convex.

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# Potentials and irreducible components on $\mathbb{R}$



- Potential functions: powerful tools in dimension 1.
- $u_{\nu-\mu}(x) := \int_{\mathbb{R}} |t-x|(\nu-\mu)(dt)$.
- $u_{\nu-\mu} \geq 0$.
- For $\mathbb{P} \in \mathcal{M}(\mu,\nu)$,

$$\begin{aligned} u_{\nu-\mu}(x_0) = 0 \quad &\iff \quad \mathbb{E}^{\mathbb{P}}[|Y-x_0|] = \mathbb{E}^{\mathbb{P}}[|X-x_0|] \\ &\iff \quad \mathbb{P}[Y > x_0 | X \leq x_0] = 0. \end{aligned}$$

- Irreducible paving: $\{u_{\nu-\mu}(X) > 0\} = \underset{k\in\mathbb{N}}{\cup} ]a_k, b_k[$.
- Irreducible component $I_k :=]a_k, b_k[$ (Beiglbock-Juillet 2012).
- $X \in I_k \implies Y \in \mathrm{cl}\, I_k$, $\mathcal{M}(\mu,\nu)$-q.s.

# Link between potential and convex functions in $\mathbb{R}$



- $\mathfrak{C} := \{f : \mathbb{R}^d \to \mathbb{R},$ convex, and $(\nu - \mu)[f] \leq 1\}$.
- If $d = 1$, we have $(\nu - \mu)[f] = \int_{\mathbb{R}} \frac{1}{2} u_{\nu-\mu}(x) f''(x) dx$.
- $u_{\nu-\mu} > 0$ on $I_k$: $\mathfrak{C}$ is compact when restricted to $I_k$ for $f \in \mathfrak{C}$, as $(\nu - \mu)[f] \leq 1$.
- To get compactness we need to "anchor" the convex function: $f_k(x) = f(x) - f(x_k) - \nabla f(x_k) \cdot (x - x_k) \geq 0$ for some $x_k \in I_k$.
- Idea of doubling the variable: $\mathbf{T}f(x, y) := f(y) - f(x) - \nabla f(x) \cdot (y - x) \geq 0$.
- Tangent convex functions: $\mathcal{T}(\mu, \nu) := \mathrm{cl}\mathbf{T}(\mathfrak{C})$

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# Definition of the irreducible convex paving in $\mathbb{R}^d$

- For $\theta \in \mathcal{T}(\mu,\nu)$, and $\mathbb{P} \in \mathcal{M}(\mu,\nu)$:

$$\mathbb{E}^{\mathbb{P}}[\theta(X,Y)] \leq 1.$$

- $Y \in \mathrm{dom}\theta(X,\cdot)$, $\mathcal{M}(\mu,\nu)-$q.s.
- $\{\mathrm{dom}\theta(x,\cdot), x \in \mathbb{R}^d\}$ is a partition of $\mathbb{R}^d$.
- For $K$ convex $G(K) := \dim(K) + g_K(K)$, with $g_K$ Gaussian measure on $\mathrm{Aff}(K)$.
- $G$ in increasing and bounded.
- Consider the minimization problem

$$\inf\Big\{\mathbb{E}^{\mu}[G(K)] : \ K := K(X) = \mathrm{dom}\theta(X,\cdot) \\ \text{for some } \theta \in \mathcal{T}(\mu,\nu)\Big\},$$

# Properties of the irreducible convex paving in $\mathbb{R}^d$





## Theorem

*There is a $\mu$-a.s. unique optimizer $I : \mathbb{R}^d \to \overset{\circ}{\mathcal{K}}$ to the optimization problem, called irreducible convex paving map. Moreover, we have the following properties: $I$ is universally measurable, $\{I(x), x \in \mathbb{R}^d\}$ is a partition of $\mathbb{R}^d$ with $X \in I(X)$, and*

$$Y \in \operatorname{cl} I(X), \quad \mathcal{M}(\mu, \nu) - q.s.$$

# Setwise duality



## Proposition

*We may find a probability measure* $\hat{\mathbb{P}} \in \mathcal{M}(\mu,\nu)$ *and* $\hat{\theta} \in \mathcal{T}(\mu,\nu)$ *such that for any* $\mathbb{P} \in \mathcal{M}(\mu,\nu)$ *and* $\theta \in \mathcal{T}(\mu,\nu)$*,*

$$\begin{aligned} \operatorname{supp} \mathbb{P}_X \subset \quad & \operatorname{conv}\left(\operatorname{supp} \hat{\mathbb{P}}_X\right) \\ & = \operatorname{cl} I(X) \\ & = \operatorname{cl} \operatorname{dom} \hat{\theta}(X,\cdot) \subset \quad \operatorname{cl} \operatorname{dom} \theta(X,\cdot), \\ & \mu - a.s. \end{aligned}$$

# Characterization of the $\mathcal{M}(\mu,\nu)$-polar sets



We also have the following characterization of $\mathcal{M}(\mu,\nu)-$polar sets. We denote by $\mathcal{N}_\mu$ and $\mathcal{N}_\nu$ the collection of all negligible sets for $\mu$ and $\nu$, respectively.

## Proposition

*A subset $N \in \mathcal{B}(\Omega)$, is $\mathcal{M}(\mu,\nu)-$polar if and only if*

$$N \subset \{X \in N_\mu\} \cup \{Y \in N_\nu\} \cup \{Y \notin J_\theta(X)\}$$

*for some $N_\mu \in \mathcal{N}_\mu$, $N_\nu \in \mathcal{N}_\nu$, and $\theta \in \mathcal{T}(\mu,\nu)$.*

Where $J_\theta(X) := I(X) \cup \mathrm{dom}\theta(X,\cdot) \cap \mathrm{cl}\, I(X)$.

# Questions ?


Figure: An example of Optimal Transport in practice.