Cover's universal portfolio, stochastic portfolio theory and the numéraire portfolio

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Advances in Financial Mathematics, Paris
January, 11, 2017

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- This questions stems from the fact that both theories ask for preference-free and general recipes how to choose a good (at least in the long run) long only portfolio among d assets.

Cover's universal portfolio - Overview

- Cover's insight reveals the phenomenon that the "wisdom of hindsight" does not give any significant advantage as compared to a properly chosen "universal" portfolio which is constructed in a predictable way.
- The relevant optimality criterion here is the asymptotic growth rate of the portfolio

$$\lim_{T \to \infty} \frac{1}{T} \log V_T,$$

where $(V_T)_{T \in \mathbb{T}}$ denotes the wealth process and $\mathbb{T}$ stands either for $\mathbb{N}$ (discrete time) or $[0, \infty)$ (continuous time).

Toy example

- Consider discrete time and denote by $s = (s^1, \dots, s^d)_{t=0}^{\infty}$ a trajectory of stock prices taking values in $\mathbb{R}^d_{++}$.
- Fix $T \in \mathbb{N}$ and think of an investor who at time T looks back and asks which stock she should have bought at time $t = 0$ by investing her initial endowment of 1 EUR and subsequently holding the stock.
- Obvious solution to this problem: pick the stock $i \in \{1, \dots, d\}$ which maximizes the performance $\frac{s^i_T}{s^i_0}$.
- It clearly also maximizes the normalized logarithmic return

$$\frac{1}{T}[\log(s^i_T) - \log(s^i_0)] \qquad i = 1, \dots, d.$$

- The "only" problem is, of course, that we have to make our choice at time $t = 0$ instead of $t = T$.

Toy example cont.

- Remedy: simply divide at time $t = 0$ the initial endowment of 1 EUR into d portions of $\frac{1}{d}$. At time T your wealth equals

$$V_T = \frac{1}{d} \sum_{j=1}^{d} \frac{s_T^j}{s_0^j} \geq \frac{1}{d} \frac{s_T^i}{s_0^i},$$

  where again i denotes the stock which performed best during the time interval $[0, T]$.

- Passing again to normalized logarithmic returns we obtain

$$\frac{1}{T} \log(V_T) \geq \frac{1}{T} \Big[\log(s_T^i) - \log(s_0^i) - \log(d) \Big].$$

- The difference between the retrospectively chosen portfolio and the "universal portfolio" consisting of equally weighing the d stocks at time $t = 0$ can thus be estimated by $\frac{\log(d)}{T}$, which tends to zero as $T \to \infty$.

Cover's setting

- Model-free setting (no probability space) in discrete time $t \in \mathbb{N}$.
- Instead of only considering "pure" investments into one of the stocks as benchmark, Cover considers all constant rebalanced portfolio strategies:
- Let $b = (b^1, \dots, b^d)$ be a fixed element of the d- dimensional closed simplex

$$\bar{\Delta}^d = \left\{ x \in \mathbb{R}^d_+ \mid \sum_{j=1}^d x^j = 1 \right\}.$$

We denote by Δ^d the interior of the simplex.

- The corresponding portfolio wealth process $(V_t^b)_{t=0}^\infty$ is given by

$$\frac{V_{t+1}^b(s)}{V_t^b(s)} = \sum_{j=1}^d b^j \frac{s_{t+1}^j}{s_t^j}, \quad V_0^b = 1$$

for each scenario $s = ((s_t^j)_{j=1}^d)_{t=0}^\infty$ of strictly positive numbers corresponding to stock prices.

Best retrospectively chosen portfolio

- For fixed T, define V_T^* by

$$V_T^*(s) = \sup_{b \in \bar{\Delta}^d} V_T^b(s),$$

which is a function depending on the scenario $s = (s_t^1, \dots, s_t^d)_{t=0}^T$. The optimizer is denoted by b^* and is refereed to as best retrospectively chosen portfolio.

- Cover's goal was to construct a "universal" portfolio chosen in a predictable way which performs as good as $(V_T^*)_{T=0}^\infty$ asymptotically for $T \to \infty$.

Cover's universal portfolio

- For a probability measure ν on $\bar{\Delta}^d$ define portfolio weights as

$$b_T^\nu = \frac{\int_{\bar{\Delta}^d} b V_T^b d\nu(b)}{\int_{\bar{\Delta}^d} V_T^b d\nu(b)}.$$

which yield the following wealth process:

$$V_t(\nu)(s) = \int_{\bar{\Delta}^d} V_t^b(s) d\nu(b).$$

Thus, Cover's universal portfolio consists in investing the portion $d\nu(b)$ of one's wealth into the constant rebalanced portfolio with weights b.

- Note that the universal portfolio strategy at each time T is built by averaging with a sort of posterior distribution of the form

$$\nu_T(A) = \frac{\int_A V_T^b d\nu(db)}{\int_{\bar{\Delta}^d} V_T^b d\nu(b)}$$

with prior distribution ν on Δ^d and the wealth at time T interpreted as likelihood function.

Cover's original result

Theorem (Cover (91))

Let ν be a probability measure on $\bar{\Delta}^d$ with full support. Then

$$\lim_{T\to\infty} \frac{1}{T} \left(\log(V_T^*(s)) - \log(V_T(\nu)(s))\right) = 0$$

for all trajectories $s = (s_t^1, \ldots, s_t^d)_{t=0}^{\infty}$ for which there are constants $0 < c \leq C < \infty$ such that

$$c \leq \frac{s_{t+1}^j}{s_t^j} \leq C, \qquad \textit{for all} \quad j = 1, \ldots, d \quad \textit{and all} \quad t \in \mathbb{N}.$$

Extensions, Improvements and an (incomplete) literature overview

- Quantitative estimates when the distribution ν is specified.
- Relaxation of the boundedness of the price relatives (Cover and Ordentlich (96), Blum and Kalai (99)): For the uniform distribution on $\bar{\Delta}^d$, they obtain

$$(\log(V_T(\nu)(s)) - \log(V_T^*(s))) \geq -(d-1)\log(T-1).$$

- Similar results in continuous time by Jamshidian (1992) for diffusions.
- Other parametric families instead of the constantly rebalanced one.
- Recent results by Wong (2015) on the nonparametric family of long only functionally generated portfolios of SPT in discrete time.
- Recently universal portfolio strategies have been studied extensively in an algorithmic and machine learning framework (Hazan and Kale (2015)).

The setting of stochastic portfolio theory

- Stochastic portfolio theory (SPT) is a theory for analyzing stock market structures and portfolio behavior and was introduced R. Fernholz.
- Main quantity of interest: Relative performance with respect to the market portfolio.
- Economically speaking, this amounts to take the market portfolio $\sum_{j=1}^{d} s^j$ as numéraire.
- One associates to the stock prices $(s^1, \ldots, s^d) \in \mathbb{R}^d_{++}$ the vector of market weights $(\mu^1, \ldots, \mu^d) \in \Delta^d$ by normalizing by the total market capitalization $\sum_{j=1}^{d} s^j$ i.e.

$$(\mu^1, \ldots, \mu^d) = \left(\frac{s^1}{\sum_{j=1}^{d} s^j}, \ldots, \frac{s^d}{\sum_{j=1}^{d} s^j} \right).$$

Portfolio maps and relative wealth processes

- A long only portfolio map is a measurable function

$$\pi : \Delta^d \to \bar{\Delta}^d$$

  which associates to the current market weights $\mu_t = (\mu_t^1, \ldots, \mu_t^d)$ the weights $(\pi(\mu_t) = (\pi^1(\mu_t), \ldots, \pi^d(\mu_t))$ corresponding to the proportion of current wealth invested in the i^{th} asset.

- The constant rebalanced portfolio strategies correspond to the constant functions $\pi : \Delta^d \to \bar{\Delta}^d$.

- Relative wealth process: $Y^\pi = \frac{V^\pi}{V^\mu}$
  - Discrete time (model-free): $\frac{Y^\pi_{t+1}}{Y^\pi_t} = \sum_{j=1}^d \pi^j(\mu_t) \frac{\mu^j_{t+1}}{\mu^j_t}$
  - Continuous time (at least in a semimartingale setting) : $\frac{dY^\pi_t}{Y^\pi_t} = \sum_{j=1}^d \pi^j(\mu_t) \frac{d\mu^j_t}{\mu^j_t}$

Program of the remainder of this talk

- Consider instead of the constantly rebalanced portfolios larger classes of non-parametric portfolio maps $\pi : \Delta^d \to \bar{\Delta}^d$.
- Compare
  1. the best retrospectively chosen portfolio in this class of portfolio maps;
  2. the analog of Cover's universal portfolio in this setting;
  3. the log-optimal portfolio within this class portfolio maps.
- Establish equal asymptotic growth rates for ergodic Markovian models for the market weights μ both in discrete and continuous time.
- "Modelfree" setup for comparing (1) and (2)

“Modelfree” setup for comparing (1) and (2) - Ingredients

Assumptions A

- $(\mathcal{G}, \|\cdot\|)$: compact set of functions generating the set of portfolio maps

$$\mathcal{FG} = \{\pi : \Delta^d \to \bar{\Delta}^d, x \mapsto \pi(x) = \Pi(G)(x) \,|\, G \in \mathcal{G}\}$$

  via some function $\Pi : \mathcal{G} \to \mathcal{FG}$. The relative wealth corresponding to $\Pi(G)$ is denoted by $Y^{\Pi(G)} = Y^G$.

- For $G \in \mathcal{G}$ and a given trajectory $(\mu_t)_{t\in\mathbb{T}}$ taking values in Δ^d, the wealth process $(Y_t^G)_{t\in\mathbb{T}}$ can be defined in a pathwise way, which is of course only an issue in continuous time.

- For every T, $G \mapsto Y_T^G$ is continuous.

- ν: Borel probability measure with full support on $\mathcal{G}$

"Modelfree" setup for comparing (1) and (2) - Types of portfolios

- The best retrospectively chosen portfolio: Define for each T and a given trajectory $(\mu_t)_{t\in\mathbb{T}}$
$$Y_T^* = \sup_{G\in\mathcal{G}} Y_T^G.$$
By compactness of $\mathcal{G}$ and continuity of $G \mapsto Y_T^G$ the optimizer exists and is denoted by G_T^*.
- The universal portfolio: Define $\widetilde{\nu} = \Pi_*\nu$ and
$$\pi_T^\nu = \frac{\int_{\mathcal{FG}} \pi(\mu_T) Y_T^\pi d\widetilde{\nu}(\pi)}{\int_{\mathcal{FG}} Y_T^\pi d\widetilde{\nu}(\pi)},$$
so that the relative wealth achieved by investing according to π^ν is given by
$$Y_T(\nu) = \int_{\mathcal{G}} Y_T^G d\nu(G).$$

Comparison between the best retrospectively and universal portfolio

The analog of Cover's Theorem reads in the present setting as follows:

Theorem (C., Schachermayer, Wong (2016))

Fix a Borel probability measure ν with full support on $\mathcal{G}$ and consider a trajectory $(\mu_t)_{t \in \mathbb{T}}$ taking values in Δ^d. Suppose that Assumptions A holds and that for every $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\frac{1}{T} \log(Y_T^G) \geq \frac{1}{T} \log(Y_T^*) - \varepsilon$$

for all $T \in \mathbb{T}$ and $G \in \mathcal{G}$ such that $\|G - G_T^*\| \leq \delta$. Then

$$\lim_{T \to \infty} \frac{1}{T} (\log(Y_T^*) - \log(Y_T(\nu))) = 0.$$

Portfolio maps in discrete time

One possible choice for $\mathcal{G} \equiv \mathcal{FG}$ in discrete time is the following set of functions:

- $\mathcal{L}^M$: set of all Lipschitz functions $\Delta^d \to \bar{\Delta}^d_{M^{-1}}$ with Lipschitz constant $M > 0$, which pertains, e.g., to the metric defined by the norm $\|\cdot\|_1$ on $\bar{\Delta}^d$.
- Here $\bar{\Delta}^d_\epsilon$ denotes the set of $p \in \Delta^d$ verifying $p^j \geq \frac{\epsilon}{d}$, for $j = 1, \dots, d$.

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Corollary (C., Schachermayer, Wong (2016))

Fix a Borel probability measure ν with full support on $\mathcal{L}^M$. For every individual sequence $(\mu_t)_{t=0}^\infty$ in Δ^d we have

$$\lim_{T\to\infty} \frac{1}{T}(\log(Y_T^{*,M}) - \log(Y_T^M(\nu))) = 0.$$

Portfolio maps in continuous time - functionally generated portfolios

We consider the following set of concave functions for some fixed $M > 0$ and $0 < \alpha \leq 1$,

$$\mathcal{G}^{M,\alpha} = \{ G \in C^{2,\alpha}(\bar{\Delta}^d), \text{concave such that} \|G\|_{C^{2,\alpha}} \leq M \text{ and } G \geq \frac{1}{M} \},$$

where $C^{2,\alpha}(\bar{\Delta}^d)$ denotes the Hölder space of 2-times differentiable functions from $\bar{\Delta}^d \to \mathbb{R}$ whose derivatives are α-Hölder continuous. That is,

$$C^{2,\alpha}(\bar{\Delta}^d) = \{ G \in C^2(\bar{\Delta}^d) \,|\, \|G\|_{C^{2,\alpha}} < \infty \},$$

where

$$\|G\|_{C^{2,\alpha}} = \max_{|\mathbf{k}| \leq 2} \|D^{\mathbf{k}} G\|_\infty + \max_{|\mathbf{k}|=2} \sup_{x \neq y} \frac{|D^{\mathbf{k}} G(x) - D^{\mathbf{k}} G(y)|}{\|x - y\|^\alpha}$$

with $\mathbf{k}$ denoting a multiindex.

Functionally generated portfolios as in SPT cont.

Lemma

Let $M, \alpha > 0$ be fixed. Then the set $\mathcal{G}^{M,\alpha}$ is compact with respect to $\| \cdot \|_{C^{2,0}}$.

Functionally generated portfolios as in SPT cont.

Lemma

Let $M, \alpha > 0$ be fixed. Then the set $\mathcal{G}^{M,\alpha}$ is compact with respect to $\| \cdot \|_{C^{2,0}}$.

To the set of generating functions $\mathcal{G}^{M,\alpha}$ we associate now the set of functionally generated portfolios $\mathcal{FG}^{M,\alpha}$ defined via

$$\mathcal{FG}^{M,\alpha} = \Bigg\{ \pi : \Delta^d \to \bar{\Delta}^d, \\ x \mapsto \pi^i(x) = x^i \left(\frac{D^i G(x)}{G(x)} + 1 - \sum_{j=1}^{d} \frac{D^j G(x)}{G(x)} x^j \right), \mid G \in \mathcal{G}^{M,\alpha} \Bigg\}.$$

The "modelfree" Master equation

Under the assumption

Assumption ((QV))

The path μ admits a continuous S_d^+-valued quadratic variation $[\mu]$ along $(\mathbb{T}_n)$ in the sense of Foellmer, i.e., for any $1 \leq i, j \leq d$ and all $t \geq 0$ the sequence

$$\sum_{s \in \mathbb{T}_n, s \leq t} (\mu^i_{s'} - \mu^i_s)(\mu^j_{s'} - \mu^j_s)$$

converges to a finite limit, denoted $[\mu^i, \mu^j]_t$, such that $t \mapsto [\mu^i, \mu^j]_t$ is continuous.

and by applying Foellmer's functional Itô calculus we get the following pathwise version of Fernholz's master equation, which also follows from Schied et al:

$$Y^G_T = \frac{G(\mu_T)}{G(\mu_0)} e^{\mathfrak{g}([0,T])}, \quad 0 \leq T < \infty,$$

where $\mathfrak{g}(dt) = -\frac{1}{2G(\mu_t)} \sum_{i,j} D^{ij} G(\mu_t) d[\mu^i, \mu^j]_t$.

Comparison between best retrospectively and universal portfolio

The analog of Cover's Theorem reads in the present setting as follows:

Corollary (C., Schachermayer, Wong (2016))

Let $M, \alpha > 0$ be fixed and let $(\mu_t)_{t \geq 0}$ be a continuous path satisfying Assumption (QV) such that for all $i \in \{1, \dots, d\}$

$$\lim_{T \to \infty} \frac{1}{T}[\mu^i, \mu^i]_T < \infty.$$

Consider a probability measure ν on $\mathcal{G}^{M,\alpha}$ with full support. Then

$$\lim_{T \to \infty} \frac{1}{T}(\log Y_T^{*,M,\alpha} - \log Y_T^{M,\alpha}(\nu)) = 0.$$

Comparison with the log-optimal portfolio

- By definition the log-optimal portfolio requires a stochastic model for μ.

Assumption ((D) - Discrete time)

The process μ is a time homogeneous, ergodic Markov process on Δ^d.

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Assumption ((D) - Discrete time)

The process μ is a time homogeneous, ergodic Markov process on Δ^d.

Assumption ((C))

The process μ is a time-homogeneous ergodic Markovian Itô-diffusion on Δ^d of the form

$$\mu_t = \mu_0 + \int_0^t c(\mu_s)\lambda(\mu_s)dt + \int_0^t \sqrt{c(\mu_s)}dW_s, \quad \mu_0 \in \Delta^d,$$

where W denotes a d-dimensional Brownian motion, λ a measurable function from $\Delta^d \to \mathbb{R}^d$, c a measurable function from $\Delta^d \to S_+^d$, and the following integrability condition $\int_0^T \lambda^\top(\mu_t)c(\mu_t)\lambda(\mu_t)dt < \infty \quad \mathbb{P}$-a.s.

The log optimal portfolio

- The log-optimal portfolio in the set $\mathcal{L}^M$ ($\mathcal{G}^{M,\alpha}$ respectively). Define the log-optimal portfolio among $\mathcal{L}^M$ ($\mathcal{G}^{M,\alpha}$ respectively) by

$$\hat{Y}_T^M = \sup_{\pi \in \mathcal{L}^M} \mathbb{E}[\log(Y_T^\pi)], \quad \text{and} \quad \hat{Y}_T^{M,\alpha} = \sup_{G \in \mathcal{G}^{M,\alpha}} \mathbb{E}[\log(Y_T^G)],$$

respectively. Note that the optimizer does not depend on T due to the time homogenous Markov property of μ.

- The global long only log-optimal portfolio: The global log-optimal portfolio over all long only strategies is defined analogously and denoted via $\hat{Y}$.

Equality of the asymptotic performance - discrete time

Theorem (C., Schachermayer, Wong (2016))

Let $\mu = (\mu_t)_{t=0}^{\infty}$ be a Δ^d-valued stochastic process satisfying Assumption D and the assumption of a finite expected yield of the log-optimal portfolio. Then we have the following equality $\mathbb{P}$-a.s.

$$\liminf_{T\to\infty} \frac{1}{T} \log(Y_T^{*,M}) = \liminf_{T\to\infty} \frac{1}{T} \log(Y_T^M(\nu)) = \lim_{T\to\infty} \frac{1}{T} \log(\hat{Y}_T^M).$$

In addition, the first equality holds true, for all sequences $(\mu_t(\omega))_{t=0}^{\infty}$ in Δ^d.

Remark

Due to the assumption of ergodicity the above asymptotic growth rates are equal to a constant. This can be weakened as long as $\lim_{T\to\infty} \frac{1}{T} \log(\hat{Y}_T^M)$ exists and some integrability conditions are satisfied.

Equality of the asymptotic performance - discrete time

To formulate a result not depending explicitly on the constant M, we define a universal portfolio $\hat{Y}(\nu) = (\hat{Y}_t(\nu))_{t=0}^{\infty}$ in the following way. For $M = 1, 2, 3, \dots$ choose a measure ν^M on $\mathcal{L}^M$ with full support. Define $\nu = \sum_{M=1}^{\infty} 2^{-M} \nu^M$ and the process $Y(\nu)$ by

$$Y_t(\nu) = \int_{\bigcup_{M=1}^{\infty} \mathcal{L}^M} Y_t^{\pi} d\nu(\pi), \qquad t \in \mathbb{N}.$$

Corollary

Under the assumptions of the above Theorem we have $\mathbb{P}$-a.s.

$$\lim_{M\to\infty} \lim_{T\to\infty} \frac{1}{T} \log(Y_T^{*,M}) = \lim_{T\to\infty} \frac{1}{T} \log(Y_T(\nu)) = \lim_{T\to\infty} \frac{1}{T} \log(\hat{Y}_T).$$

Equality of the asymptotic growth rates

Theorem (C., Schachermayer, Wong (2016))

Let $M, \alpha > 0$ be fixed and let $(\mu_t)_{t \geq 0}$ be a stochastic process satisfying Assumption (C). Moreover, suppose that

$$\int_{\Delta^d} c^{ii}(x)\rho(dx) < \infty, \quad \textit{for all } i \in \{1, \dots, d\},$$

$$\int_{\Delta^d} \max_{i \in \{1,\dots,d\}} |(c(x)\lambda(x))^i|\rho(dx) < \infty.$$

Then

$$\liminf_{T\to\infty} \frac{1}{T} \log Y_T^{*,M,\alpha} = \liminf_{T\to\infty} \frac{1}{T} \log Y_T^{M,\alpha}(\nu) = \lim_{T\to\infty} \frac{1}{T} \log \hat{Y}_T^{M,\alpha}, \quad \mathbb{P}\text{-}a.s.$$

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Then

$$\liminf_{T \to \infty} \frac{1}{T} \log Y_T^{*,M,\alpha} = \liminf_{T \to \infty} \frac{1}{T} \log Y_T^{M,\alpha}(\nu) = \lim_{T \to \infty} \frac{1}{T} \log \hat{Y}_T^{M,\alpha}, \quad \mathbb{P}\text{-a.s.}$$

A result independent of M yielding equality with the global log optimal portfolio can be achieved whenever it is functionally generated by some concave function $G \in C^2(\bar{\Delta}^d)$.

Conclusions

- Establish a link between stochastic portfolio theory and Cover's universal portfolio by replacing Cover's constantly rebalanced portfolios by more general portfolio maps $\pi : \Delta^d \mapsto \bar{\Delta}^d$.
- This yields equality of the asymptotic growth rates of
  1. the best retrospectively chosen portfolio;
  2. the analog of Cover's universal portfolio in this setting;
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  for ergodic Markovian models for the market weights μ, both in discrete and continuous time.

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Thank you for your attention!