Robust Hedging of Options on a Leveraged Exchange Traded Fund

Alexander M. G. Cox   Sam M. Kinsley

University of Bath

Recent Advances in Financial Mathematics, Paris, 10th January, 2017

Model-independent/Robust bounds for option prices

- Aim: make statements about the price of options given very mild modelling assumptions
- Incorporate market information by supposing the prices of vanilla call options are known
- Typically want to know the largest/smallest price of an exotic option (Lookback option, Barrier option, Variance option, Asian option,. . .) given observed call prices, but with (essentially) no other assumptions on behaviour of underlying
- This talk: options on *Leveraged Exchange Traded Funds (LETF)*
- Why? Heavily traded, and interesting features to the solution!

Financial Setting

- Option priced on an asset $(S_t)_{t\in[0,T]}$, option payoff $F((S_t)_{t\in[0,T]})$
- Dynamics of S unspecified, but suppose paths are continuous, and we see prices of call options at all strikes K and at maturity time T
- Assume for simplicity that all prices are discounted — this won't affect our main results
- Under risk-neutral measure, S should be a (local-)martingale, and we can recover the law of S_T at time T, μ say, from call prices $C(K)$

Leveraged Exchange Traded Fund (LETF)

- ETF attempts to match returns on a benchmark asset/index 1:1
- LETF attempts to match returns on a benchmark asset/index up to factor, e.g. 2:1 — 10% increase in index → 20% increase in LETF
- Over time, e.g. daily rebalancing leads to tracking errors
- Dynamics of the LETF with leverage ratio $\beta > 1$ are given by

$$L_t = S_t^\beta \exp\left(-\frac{\beta(\beta-1)}{2}V_t\right),$$

  V_t is the accumulated quadratic variation of $\log S_t$

- Eliminate V_t by time change, $\tau_t := \inf\{s \geq 0 : \ V_s = t\}$ and $X_t := S_{\tau_t}$. So,

$$\mathrm{d}\langle X\rangle_t = \mathrm{d}\langle S\rangle_{\tau_t} = S_{\tau_t}^2 \mathrm{d}V_{\tau_t} = X_t^2 \mathrm{d}t$$

  and X_t is a geometric Brownian motion (GBM)

LETF model-independent pricing problem

Want to consider (maximum) price of call option on LETF under assumption that law of S_T (under $\mathbb{Q}$) is known, but no other modelling assumption. Corresponds to:

Main Problem

Find

$$\sup_{\tau} \mathbb{E}\left[\left(X_\tau^\beta \exp\left(-\frac{\beta(\beta-1)}{2}\tau\right) - k\right)_+\right], \qquad \text{(LOptSEP)}$$

over stopping times τ such that $X_\tau \sim \mu$, where X is a GBM

- Also: is there an arbitrage if the price of the option on the LETF exceeds this?
- This is a form of Optimal Skorokhod Embedding Problem (OptSEP)

Existing Literature

Rich literature on these problems:

- Starting with Hobson ('98) connection with Skorokhod Embedding problem $\rightarrow$ explicit optimal solutions for many different payoff functions (Brown, C., Dupire, Henry-Labordère, Hobson, Klimmek, Obłój, Rogers, Spoida, Touzi, Wang,...)
- Recently, model-independent duality has been proved by Dolinsky-Soner ('14):

$$\sup_{\mathbb{Q}:S_T\sim\mu} \mathbb{E}^{\mathbb{Q}}[X_T] = \text{price of cheapest super-replication strategy}$$

  Here the super-replication strategy will use both calls and dynamic trading in underlying, and is model-independent. The sup is taken over measures $\mathbb{Q}$ for which S is a martingale. (See also Hou-Obłój and Beiglböck-C.-Huesmann-Perkowski-Prömel)
- The problem of finding the maximising martingale S is commonly called the Martingale Optimal Transport problem (MOT)

Geometric Characterisation via BCH

- Key observation of Beiglböck, C., Huesmann [BCH] (2016):

  Solutions to (OptSEP) are often characterised by simple geometric criteria

- Geometric criteria typically determined by the monotonicity principle ([BCH]):

  if I am better off 'stopping' a currently running path, and 'transplanting' the tail onto another stopped path (stopping at the same level), my solution is not optimal

- Monotonicity principle can be used to show that optimisers of (OptSEP) have a certain geometric form

Form of Optimiser: K-cave barrier

- Recall, problem is to maximise $\mathbb{E}[(M_\tau - k)_+]$, where $M_t = X_t^\beta \mathrm{e}^{-\beta(\beta-1)t/2}$ is a martingale. Intuitively, aim to maximise local time of M at k
- Can compute $M_t = k$ when $K(X_t) = t$, $K(x) = \frac{2}{\beta(\beta-1)} \ln(\frac{x^\beta}{k})$
- A K-cave barrier is a subset $\mathcal{R}$ of $\mathbb{R}_+ \times \mathbb{R}_+$ of the form $\mathcal{R} = \{(t,x) : t \leq \ell(x) \text{ or } t \geq r(x)\}$, where $\ell(x) \leq K(x) \leq r(x)$
- Similar concept (cave barrier) appeared in [BCH] ($K = const$)

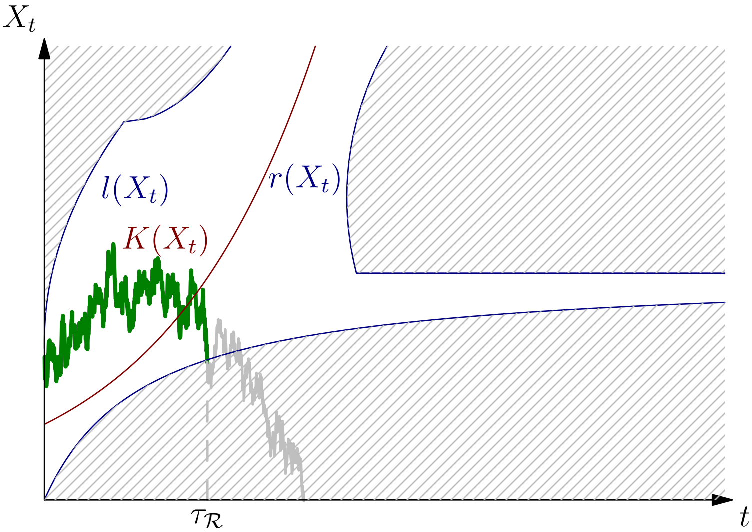
Theorem

There exists an optimiser to (LOptSEP) which is of the form

$$\tau_{\mathcal{R}} := \inf\{t > 0 : (t, X_t) \in \mathcal{R}\}$$

where $\mathcal{R}$ is a K-cave barrier.

K-cave barriers



(Non-)uniqueness of Barriers

- *Normally*, at this point, a simple argument essentially due to Loynes would imply that there is a unique K-cave barrier with the right stopping distribution, which would then be the optimiser.
- However, for the K-cave barriers, there are generally multiple K-cave barriers which embed the same distribution; consider 3-atom measures. Crucial question:

How to identify the optimal K-cave barrier?

PDE Heuristics for the Dual Solution

- We expect the Dual solution (superhedging portfolio) to take the form: $\exists G, \lambda$ such that

$$G(t,x) + \lambda(x) \geq F(t,x),$$

where λ represents a portfolio of calls, F is the payoff of the option, and γ is the proceeds of a dynamic trading strategy in the underlying.
- We argue heuristically, inspired by arguments of Henry-Labordère: write $F^\lambda(t,x) = F(t,x) - \lambda(x)$. Then we require:

$$\mathcal{L}G := \frac{x^2}{2}\partial_x^2 G + \partial_t G \leq 0 \text{ and } G \geq F^\lambda \qquad \forall(t,x)$$

and expect equality in PDE in $\mathcal{R}^\complement$, and $G = F^\lambda$ in $\mathcal{R}$.
- Also conjecture smooth fit: $\partial_t G = \partial_t F^\lambda = \partial_t F$ on boundaries

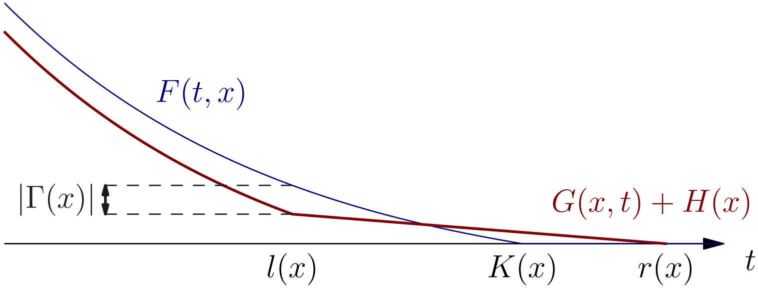
$$\implies M := \partial_t G \text{ solves } \mathcal{L}M = 0 \text{ in } \mathcal{R}^\complement \text{ and } M = \partial_t F \text{ on } \partial\mathcal{R}$$

PDE Heuristics for the Dual Solution

- In particular, we get: $M(t,x) = \mathbb{E}^{(t,x)}[\partial_t F(X_{\tau_{\mathcal{R}}}, \tau_{\mathcal{R}})]$, and integrating, we see that
$$G(t,x) = \int_t^{r(x)} M(s,x)\,\mathrm{d}s - Z(x)$$
for some function Z.
- In fact, Z can be chosen (uniquely up to affine functions) in such a way to make G a martingale in $\mathcal{R}^{\complement}$.
- Now $G(t,x) \geq F^{\lambda}(t,x)$ at $t = \ell(x), t = r(x)$ implies that:

$$\lambda(x) \geq Z(x) + \max\{\underbrace{\quad 0 \quad}_{t=r(x)}, \underbrace{F(\ell(x),x) - \int_{\ell(x)}^{r(x)} M(s,x)\,\mathrm{d}s}_{\substack{:=\Gamma(x) \\ t=\ell(x)}}\}$$

Γ-condition: $\Gamma > 0$



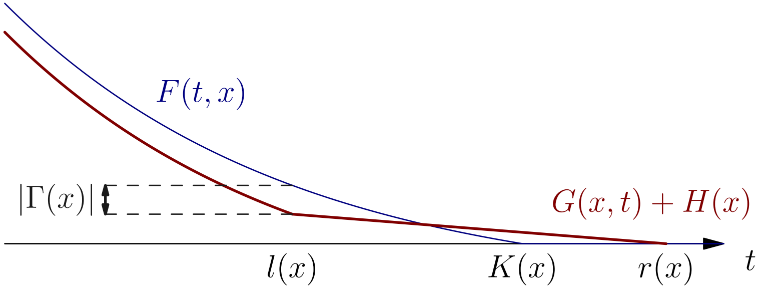
Lemma (Easy)

Suppose $\mathcal{R}$ is a K-cave barrier which embeds μ, and such that $\Gamma(x) = 0$ for all x. Then $\tau_{\mathcal{R}}$ is an optimiser of (LOptSEP).

Theorem (Hard)

There exists a K-cave barrier $\mathcal{R}$ which embeds μ, and such that $\Gamma(x) = 0$ for all x.

Γ-condition: $\Gamma > 0$



Lemma (Easy)

Suppose $\mathcal{R}$ is a K-cave barrier which embeds μ, and such that $\Gamma(x) = 0$ for all x. Then $\tau_{\mathcal{R}}$ is an optimiser of (LOptSEP).

Theorem (Hard)

There exists a K-cave barrier $\mathcal{R}$ which embeds μ, and such that $\Gamma(x) = 0$ for all x.

Dual Feasibility

First step to proving the results:

Theorem

The dual solution described above is indeed a dual solution (i.e. G is a martingale for some suitable Z).

- Shown using essentially probabilistic techniques
- NB: No ‘explicit’ form for Z
- Clearly $\Gamma = 0$ is then a sufficient condition $\implies$ primal = dual
- But: Not enough for theorem... Know (e.g. Dolinsky & Soner) that no duality gap, but don't know optimal dual solution of the form above

Discretisation of Problem

- Idea: Take the original problem, and discretise time and space suitably (Random walk converging to the original Brownian motion)
- Can formulate the original problem in the discrete setting and formulate (LOptSEP) as a (countably infinite) linear programming problem
- Strong duality holds (in an appropriate sense) for the discretised problem, and can show existence of dual solutions, and natural condition corresponding to $\Gamma = 0$
- In the limit, exists optimal barrier, and embedding, and can make sense of $\Gamma = 0$ condition

$$\implies \textit{Theorem holds}$$

Conclusions

- Formulated the model-independent pricing problem for call on a Leveraged Exchange Traded Fund
- Corresponds to an interesting form of embedding problem: geometric characterisation does not guarantee uniqueness
- Need an additional condition, based on dual solution to determine optimal stopping region
- Proof of optimiser based on discretisation and