Mean Field Games of Timing

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SOURCE

talk based on joint work with **F. Delarue** & **D. Lacker**

R.C. & F. Delarue: Probabilistic Theory of Mean Field Games

- vol. I, Mean Field FBSDEs, Control, and Games.
- vol. II, Mean Field Games with Common Noise and Master Equations.

Stochastic Analysis and Applications. Springer Verlag, 2017.

Economic Models of Illiquidity & Bank Runs

- **Bryant** ('80) **Diamond-Dybvig** ('83, depositor insurance)
  - ***Bank Runs***, deterministic, static, *undesirable* equilibrium
- **Morris-Shin** ('03,'04)
  - short horizon traders, ***Liquidity Black Holes***, investors' *private* (noisy) signals
- **Rochet-Vives** ('04)
  - still static, investors' *private* (noisy) signals, lender of last resort
- **He-Xiong** ('09) **Minca-Wissel** ('15, '16)
  - **dynamic** continuous time model, **perfect observation**
  - exogenous randomness for **staggered** debt maturities
  - investors choose **to roll** or **not to roll**
- **O. Gossner's lecture** ('14) : first game of timing
  - diffusion model for the value of assets of the bank
  - investors have private noisy signals
  - investors choose a time to withdraw funds
- **M. Nutz** ('16) Toy model for MFG game of timing with a continuum of players

Continuous Time Bank Run Model

Inspired by **Gossner**'s lecture

- N depositors
- Amount of each individual (initial & final) deposit $D_0^i = 1/N$
- Current interest rate r
- Depositors promised return $\bar{r} > r$
- Y_t = value of the assets of the bank at time t,
- Y_t Itô process, $Y_0 \geq 1$
- $L(y)$ liquidation value of bank assets if $Y = y$
- Bank has a credit line of size $L(Y_t)$ at time t at rate $\bar{r}$
- Bank uses credit line each time a depositor runs (withdraws his deposit)

Bank Run Model (cont.)

- Assets mature at time T, no transaction after that
- If $Y_T \geq 1$ every one is paid in full
- If $Y_T < 1$ **exogenous default**
- **Endogenous default** at time $t < T$ if depositors try to withdraw **more** than $L(Y_t)$

Bank Run Model (cont.)

Each depositor $i \in \{1, \cdots, N\}$

- has access to a **private signal** X_t^i at time t

$$dX_t^i = dY_t + \sigma dW_t^i, \qquad i = 1, \cdots, N$$

- **chooses a time** $\tau^i \in \mathcal{S}^{X^i}$ at which to **TRY** to withdraw his deposit
- collects **return** $\bar{r}$ until time τ^i
- tries to **maximize**

$$J^i(\tau^1, \cdots, \tau^N) = \mathbb{E}\Big[g(\tau^i, Y_{\tau^i})\Big]$$

where

- $g(t, Y_t) = e^{(\bar{r}-r)t\wedge\tau}(L(Y_t) - N_t/N)^+ \wedge \frac{1}{N}$
- N_t number of withdrawals before t
- $\tau = \inf\{t;\ L(Y_t) < N_t/N\}$

Bank Run Model: case of full information

Assume

- $\sigma = 0$, i.e. Y_t is **public knowledge** !
- the function $y \hookrightarrow L(y)$ is also public knowledge
- $\tau^i \in \mathcal{S}^Y$

In **ANY** equilibrium

$$\tau^i = \inf\{t;\ L(Y_t) \leq 1\}$$

- Depositors withdraw at the **same time** (**run on the bank**)
- Each depositor gets his deposit back (**no one gets hurt!**)

Highly Unrealistic

Depositors should **wait longer** because of **noisy private signals**

Games of Timing

N players, states (observations / private signals) X_t^i at time t

$$dX_t^i = dY_t + \sigma dW_t^i$$

Y_t common unobserved signal (Itô process)

$$dY_t = \mu_t dt + \sigma_t dW_t^0$$

Each player maximizes

$$J^i(\tau^1, \cdots, \tau^N) = \mathbb{E}\Big[g(\tau^i, X_{\tau^i}, Y_{\tau^i}, \overline{\mu}^N([0, \tau^i])\Big]$$

where

- each τ^i is a $\mathcal{F}^{X^i}$ stopping time
- $\overline{\mu}^N = \frac{1}{N}\sum_{i=1}^N \delta_{\tau^i}$ empirical distribution of the τ^i's
- $g(t, x, y, p)$ is the reward to a player for
  - exercising his timing decision at time t when
  - his private signal is $X_t^i = x$,
  - the unobserved signal is $Y_t = y$,
  - the proportion of players who already exercised their right is p.

Abstract MFG Formulation

Recall

$$\begin{cases} dY_t = b_t dt + \sigma_t dW_t^0 \\ dX_t = dY_t + \sigma dW_t, \end{cases}$$

More generally:

1. The **states** of the players are given by a single measurable function

$$X : \mathcal{C}([0,T]) \times \mathcal{C}([0,T]) \mapsto \mathcal{C}([0,T])$$

   progressively measurable $X(w^0, w)_t$ depends only upon $w^0_{[0,t]}$ and $w_{[0,t]}$,

2. $X^i = X(W^0, W^i)$ state process for player i

3. **Reward / cost function** F on $\mathcal{C}([0,T]) \times \mathcal{C}([0,T]) \times \mathcal{P}([0,T]) \times [0,T]$ **progressively measurable** $F(w^0, w, \mu, t)$ depends only upon $w^0_{[0,t]}$, $w_{[0,t]}$, and $\mu([0,s])$ for $0 \le s \le t$, e.g.

$$F(\mathbf{W}^0, \mathbf{W}, \mu, t) = \exp((\bar{r} - r)t)\left[\frac{1}{N} \wedge \Big(L(Y_t) - \mu([0,t))\Big)^+\right],$$

Approximate Nash Equilibria

Definition

If $\epsilon > 0$, a set $(\tau^{1,*}, \cdots, \tau^{N,*})$ of stopping time $\tau^{i,*} \in \mathcal{S}_{X^i}$ is said to be an ϵ-Nash equilibrium if for every $i \in \{1, \cdots, N\}$ and $\tau \in \mathcal{S}_{X^i}$ we have:

$$\mathbb{E}[F(W^0, W^i, \overline{\mu}^{N,-i}, \tau^{i,*})] \geq \mathbb{E}[F(W^0, W^i, \overline{\mu}^{N,-i}, \tau)] - \epsilon,$$

$\overline{\mu}^{N,-i}$ denoting the empirical distribution of $(\tau^{1,*}, \cdots, \tau^{i-1,*}, \tau^{i+1,*}, \cdots, \tau^{N,*})$.

Weak Characterization

the set of weak limits as $N \to \infty$ of ϵ_N - Nash equilibria when $\epsilon_N \searrow 0$ coincide with the set of weak solutions of the MFG equilibrium problem

Strong Formulation of the MFG of Timing

$$J(\mu,\tau) = \mathbb{E}[F(W^0, W, \mu, \tau)]$$

Definition

A stopping time $\tau^* \in \mathcal{S}_X$ is said to be a strong MFG equilibrium if for every $\tau \in \mathcal{S}_X$ we have:

$$J(\mu,\tau^*) \geq J(\mu,\tau)$$

with $\mu = \mathcal{L}(\tau^*|W^0)$.

MFG of Timing Problem

1. *Best Response Optimization*: for each random environment $\boldsymbol{\mu}$ solve

$$\hat{\theta} \in \arg \sup_{\theta \in \mathcal{S}_X, \theta \leq T} J(\mu,\theta);$$

2. *Fixed-Point Step*: find $\boldsymbol{\mu}$ so that

$$\forall t \in [0,T],\ \mu(W^0,[0,t]) = \mathbb{P}[\hat{\theta} \leq t | W^0].$$

Weak Mean Field Equilibrium (MFE)

Probability measure P on

$$\Omega := \mathcal{C}([0,T]) \times \mathcal{C}([0,T]) \times \mathcal{P}(\mathcal{C}([0,T]) \times [0,T]) \times [0,T]$$

such that:

1. (W^0, W) is a Wiener process with respect to the full filtration $\mathbb{F}_+^{W^0,W,\mu,\tau}$.
2. (W^0, μ) is independent of W.
3. τ is **compatible** with (W^0, W, μ), in the sense that $\mathcal{F}_{t+}^{\tau}$ is conditionally independent of $\mathcal{F}_T^{W^0,W,\mu}$ given $\mathcal{F}_{t+}^{W^0,W,\mu}$, for every $t \in [0,T]$.
4. The optimality condition holds:

$$\mathbb{E}^P[F(W^0, W, \mu^\tau, \tau)] = \sup_{P'} \mathbb{E}^{P'}[F(W^0, W, \mu^\tau, \tau)],$$

   where the supremum is over all $P' \in \mathcal{P}(\Omega)$ satisfying (1-3) as well as $P' \circ (W^0, W, \mu)^{-1} = P \circ (W^0, W, \mu)^{-1}$.
5. The weak fixed point condition holds: $\mu = P\left((W, \tau) \in \cdot \mid W^0, \mu\right)$.

Sanity Check

From the above definition

Assume

- F is bounded, jointly measurable,
- $t \mapsto F(w^0, w, m, t)$ is continuous for every m and $\mathcal{W}^2$-almost every (w^0, w)
- τ^* is a strong MFE,

and define $\mu = \mathcal{W}^2(\tau^* \in \cdot | W^0)$. Then the measure

$$P = \mathcal{W}^2 \circ (W^0, W, \mu, \tau^*)^{-1}$$

is a weak MFE. where $\mathcal{W}^2$ standard Wiener measure on $\mathcal{C}([0, T]) \times \mathcal{C}([0, T])$.

Rationale for the Compatibility Condition

Working with weak limits $\Longrightarrow$ Loss of measurability

- If (Z, Y_n) converge weakly to (Z, Y)
- If Y_n is Z-measurable for each n,

No reason why Y should be a function of Z

We cannot expect τ to be (W^0, W, μ)-measurable after taking weak limits

Meaning of compatibility
One randomizes externally to the signal (W^0, W, μ), as long as at each time t this randomization is conditionally independent of all future information given the history of the signal.

Mathematically

If τ is compatible, there exists a sequence of $\mathbb{F}^{W^0,W,\mu}$-stopping times τ_k such that $(W^0, W, \mu, \tau_k) \Rightarrow (W^0, W, \mu, \tau)$

Example of a Weak Solution

Assumption

- F is bounded and jointly measurable,
- $\mathcal{P}([0,T]) \times [0,T] \ni (m,t) \mapsto F(w^0, w, m, t)$ is continuous for $\mathcal{W}^2$-almost every $(w^0, w) \in \mathcal{C}([0,T])^2$

Theorem If $\epsilon_n \searrow 0$, and $\vec{\tau}^{\,n} = (\tau_1^n, \ldots, \tau_n^n)$ is an ϵ_n-Nash equilibrium for the n-player game for each n, and

$$P_n = \frac{1}{n}\sum_{i=1}^n \mathbb{P} \circ \left(W^0, W^i, \frac{1}{n}\sum_{i=1}^n \delta_{(W^i, \tau_i^n)}, \tau_i^n \right)^{-1}.$$

Then $(P_n)_{n=1}^\infty$ is tight, and every weak limit is a weak MFE.

Theorem Let P be a weak MFE. Then there exist $\epsilon_n \to 0$ and ϵ_n-Nash equilibria $\vec{\tau}^{\,n} = (\tau_1^n, \ldots, \tau_n^n)$ such that

$$P = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \mathbb{P} \circ \left(W^0, W^i, \frac{1}{n}\sum_{i=1}^n \delta_{(W^i, \tau_i^n)}, \tau_i^n \right)^{-1}.$$

In fact, if $\tau^* = \tau^*(B, W)$ is a strong MFE in the sense of Definition **??**, then we can take $\vec{\tau}^{\,n}$ of the form $\tau_i^n = \tau^*(B, W^i)$.

Back to the Search for Strong Equilibria

Notation

$$J(\mu, \tau) = \mathbb{E}[F(W^0, W, \mu, \tau)]$$

Recall

A stopping time $\tau^* \in \mathcal{S}_X$ is said to be a strong MFG equilibrium if for every $\tau \in \mathcal{S}_X$ we have:

$$J(\mu, \tau^*) \geq J(\mu, \tau)$$

with $\mu = \mathcal{L}(\tau^* | W^0)$.

MFG of Timing Problem

1. *Best Response Optimization*: for each random environment $\boldsymbol{\mu}$ solve

$$\hat{\theta} \in \arg \sup_{\theta \in \mathcal{S}_X, \theta \leq T} J(\mu, \theta);$$

2. *Fixed-Point Step*: find $\boldsymbol{\mu}$ so that

$$\forall t \in [0, T], \ \mu(W^0, [0, t]) = \mathbb{P}[\hat{\theta} \leq t | W^0].$$

Assumptions

- **(C)** For each fixed $(w^0, w) \in \mathcal{C}([0,T]) \times \mathcal{C}([0,T])$, $(\mu, t) \mapsto F(w^0, w, \mu, t)$ is continuous.
- **(SC)** For each fixed $(w^0, w, \mu) \in \mathcal{C}([0,T]) \times \mathcal{C}([0,T]) \times \mathcal{P}([0,T])$, $t \mapsto F(w^0, w, \mu, t)$ is upper semicontinuous.
- **(ID)** For any progressively measurable random environments $\mu, \mu' : \mathcal{C}([0,T]) \mapsto \mathcal{P}([0,T])$ s.t. $\mu(w^0) \leq \mu'(w^0)$ a.s.

$$M_t = F(W^0, W, \mu'(W^0), t) - F(W^0, W, \mu(W), t)$$

  is a sub-martingale.

(ID) holds when F has **increasing differences** $t \leq t'$ and $\mu \leq \mu'$ imply:

$$F(w^0, w, \mu', t') - F(w^0, w, \mu', t) \geq F(w^0, w, \mu, t') - F(w^0, w, \mu, t).$$

(ID) $\Longrightarrow$ the expected reward J has also increasing differences

$$J(\mu', \tau') - J(\mu', \tau) \geq J(\mu, \tau') - J(\mu, \tau)$$

Major Disappointment: if$F(w^0, w, \mu, t) = G(\mu[0,t])$ for some real-valued continuous function G on $[0,1]$ which we assume to be differentiable on $(0,1)$, if F satisfies assumption **(ID)**, then G **is constant**!

Fixed Point Results on Order Lattices

Recall: A partially ordered set $(\mathcal{S}, \leq)$ is said to be a lattice if:

$$x \vee y = \inf\{z \in \mathcal{S};\ z \geq x, z \geq y\} \in \mathcal{S}$$

and

$$x \wedge y = \sup\{z \in \mathcal{S};\ z \leq x, z \leq y\} \in \mathcal{S},$$

for all $x, y \in \mathcal{S}$. A lattice $(\mathcal{S}, \leq)$ is said to be complete if every subset $S \subset \mathcal{S}$ has a greatest lower bound $\inf S$ and a least upper bound $\sup S$, with the convention that $\inf \emptyset = \sup \mathcal{S}$ and $\sup \emptyset = \inf \mathcal{S}$.

Example: The set $\mathcal{S}$ of stopping times of a right continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$

Fact 1: If $\mathcal{S}$ is a complete lattice and $\Phi : \mathcal{S} \ni x \mapsto \Phi(x) \in \mathcal{S}$ is order preserving in the sense that $\Phi(x) \leq \Phi(y)$ whenever $x, y \in \mathcal{S}$ are such that $x \leq y$, the set of fixed points of Φ is a non-empty complete lattice.

Another definition: A real valued function f on a lattice $(\mathcal{S}, \leq)$ is said to be supermodular if for all $x, y \in \mathcal{S}$

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y).$$

Existence of Strong Equilibria

Under assumptions **(SC)** and **(ID)** there exists a strong equilibrium.

- $\mathcal{S}_X$ stopping times for the filtration of X
- $\mathcal{M}_0(T)$ random $\mathbb{F}^{W_0}$-adapted probability measures on $[0, T]$

$$\mathcal{M}_T^0 \ni \mu \mapsto \Phi(\mu) = \arg\max_{\tau \in \mathcal{S}_{\mathbf{X}}} J(\mu, \tau)$$

  is nondecreasing in the strong set order
- $\Phi(\mu)$ is a nonempty complete sub-lattice of $\mathcal{S}_{\mathbf{X}}$.
- $\Phi(\mu)$ has a maximum $\phi^*(\mu)$ and a minimum $\phi_*(\mu)$
- $\phi^* : \mathcal{M}_T^0 \to \mathcal{S}_{\mathbf{X}}$ is non-decreasing
- $\psi : \mathcal{S}_{\mathbf{X}} \to \mathcal{M}_T^0$ defined by $\psi(\tau) = \mathcal{L}(\tau | W^0)$ is monotone
- $\phi^* \circ \psi$ is a monotone map from $\mathcal{S}_{\mathbf{X}}$ to itself
- Since $\mathcal{S}_{\mathbf{X}}$ is a complete lattice **Tarski's fixed point Theorem** gives a fixed point τ i.e. a strong equilibrium for the mean field game of timing

Existence of Strong Equilibria

If **(C)** holds there exist strong equilibria τ^* and θ^* such that for any strong equilibrium τ we have $\theta^* \leq \tau \leq \tau^*$ a.s.

- $\tau_0 \equiv T$,
- $\tau_i = \phi^* \circ \psi(\tau_{i-1})$ for $i \geq 1$ by induction.
- $\tau_1 \leq \tau_0$,
- If $\tau_i \leq \tau_{i-1}$, the monotonicity of $\phi^* \circ \psi$ implies $\tau_{i+1} = \phi^* \circ \psi(\tau_i) \leq \phi^* \circ \psi(\tau_{i-1}) = \tau_i$.
- Define $\tau^* = \lim_{i\to\infty} \tau_i$
- $\tau^* \in \mathcal{S}_{\mathbf{X}}$ (right continuous filtration)
- $\lim_{i\to\infty} \psi(\tau_i) = \psi(\tau^*)$
- For any $\sigma \in \mathcal{S}_{\mathbf{X}}$ $J(\psi(\tau_i), \tau_{i+1}) \geq J(\psi(\tau_i), \sigma)$
- (dominated convergence + F continuous) $\Rightarrow J(\psi(\tau^*), \tau^*) \geq J(\psi(\tau^*), \sigma)$.
- τ^* is a mean field game of timing equilibrium in the strong sense.

Existence of Strong Equilibria (cont.)

- $\theta_0 \equiv 0$,
- $\theta_i = \phi_* \circ \psi(\theta_{i-1})$ for $i \geq 1$.
- $\theta_0 \leq \theta_1$, and as before $\theta_{i-1} \leq \theta_i$.
- Define θ_* as the a.s. limit of the non-decreasing sequence of stopping times $(\theta_i)_{i \geq 1}$.
- As before $\theta_* \in \mathcal{S}_{\mathbf{X}}$ is a fixed point of the map $\phi_* \circ \psi$ and thus a strong equilibrium.

Existence of Strong Equilibria (cont.)

- If τ is any equilibrium,
- τ is a fixed point of the set-valued map $\Phi \circ \psi$

$$\tau \in \Phi(\psi(\tau))$$

- $\theta_0 = 0 \leq \tau \leq T = \tau_0$
- Apply $\phi_* \circ \psi$ and $\phi^* \circ \psi$ repeatedly to the left and right sides
- Get $\theta_n \leq \tau \leq \tau_n$ for each n,
- In the limit $\theta_* \leq \tau \leq \tau^*$.