Interest rate models enhanced with local volatility

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Introduction

Backgound:

- In fixed income world, Dupire-like local volatility does not exist.
- In [3], Gatarek et el have considered a one-dimensional Cheyette process enhanced with a local vol, and derived an (approximate) Dupire-like local vol.
- Dimensional curse for markov functionals.

Introduction

Our work:

- A general equation which impose a generic (multi-factors) interest rate model is calibrated to a strip of rolling maturity swaptions. (Constant tenor, diagonals, etc.)
- As an example: Cheyette 1-d enhanced with local vol.
- Extension to multi-dimensional models (Cheyette, BGM).

Matching a rolling maturity swaption: constant tenor

Rolling Swap rate $s_t^{t,t+\theta}$ with maturity t and tenor θ, its dynamics is:

$$ds_t^{t,t+\theta} = \underbrace{\sigma_t^{t,t+\theta}}_{\textit{target local vol}} \; dW_t^{t,t+\theta} + \cdot dt$$

Proposition (Local vol calibration condition)

$C(t,K) = C^{\text{mkt}}(t,K)$ for all (t,K) if and only if

$$\mathbb{E}^{\mathbb{Q}}[(\sigma_t^{t,t+\theta})^2 | s_t^{t,t+\theta} = K] = 2\frac{\partial_t C^{\text{mkt}}(t,K) - KC^{\text{mkt}}(t,K) + K^2 \partial_K C^{\text{mkt}}(t,K)}{\partial_K^2 C^{\text{mkt}}(t,K)}$$
$$+2\frac{\mathbb{E}^{\mathbb{Q}}[\frac{1_{s_t^{t,t+\theta}>K}}{B_t}(f_{t,t} - f_{t,t+\theta} P_{t,t+\theta})]}{\partial_K^2 C^{\text{mkt}}(t,K)} \qquad (1)$$

$f_{t,\alpha}$: forward instantaneous rate at α.
$P_{t,\alpha}$: zero coupon of maturity α.

An example: Cheyette 's model with LV

- Markovian processes:

$$dX_t = (Y_t - \lambda_t X_t)dt + \sigma_t dW_t$$
$$dY_t = (\sigma_t^2 - 2\lambda Y_t)dt$$

- zero-coupon bond:

$$\begin{aligned} P_{tT} &= \frac{P_{0T}}{P_{0t}} e^{G_{tT}X_t - \frac{1}{2}G_{tT}^2 Y_t}, \quad G_{tT} = \frac{e^{-\lambda(T-t)} - 1}{\lambda} \\ f_{tT} &\equiv -\partial_T \ln P_{tT} = f_{0T} + e^{-\lambda(T-t)}(X_t - G_{tT}Y_t) \\ r_t &= f_{0t} + X_t \end{aligned}$$

- Local volatility specification: $\sigma_t = \frac{\sigma(t, s_t^{t,t+\theta(t)})}{(\partial_X s_t^{t,t+\theta(t)})(t, X_t, Y_t)}$
- Swap rate dynamics is then:

$$ds_t^{t,t+\theta} = \partial_X s_t^{t,t+\theta(t)}(t, X_t, Y_t) \cdot dX_t = \sigma(t, s_t^{t,t+\theta(t)}) \cdot dW_t^{t,t+\theta}$$

Matching marginals

From Equation (1), $C(t,K) \equiv C^{\mathrm{mkt}}(t,K)$ for all $(t,K) \in [0,T] \times \mathbb{R}$ if and only if $\sigma(t,K)$ is given by

$$\sigma(t,K)^2 = \underbrace{\sigma_{\mathrm{loc}}(t,K)^2}_{\text{can be read from market datas}} + 2\frac{\Xi(t,K)}{\partial_K^2 C^{\mathrm{mkt}}(t,K)}$$

with

$$\Xi(t,K) \equiv \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^t r_s ds}\xi_t]$$
$$\xi_t \equiv 1_{s_t^{t,t+\theta} > K}(f_{t,t} - f_{t,t+\theta}P_{t,t+\theta})$$

Numerical solutions $\Rightarrow$ particle method:
$\Rightarrow$ *Approximate* $\Xi(t,K)$ *by empirical distributionateachstep*

Numerical examples

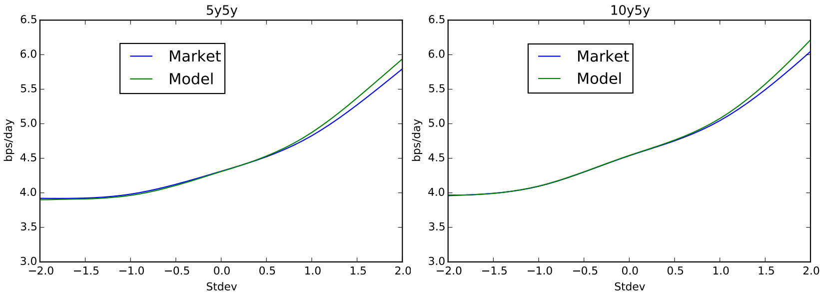

Figure: Swaption smile with maturities 5Y/10Y and tenor of 5Y compared to implied volatilities (EUR, 15-April-2016). $N = 2^{12}$ particles, 2^{15} simulations.

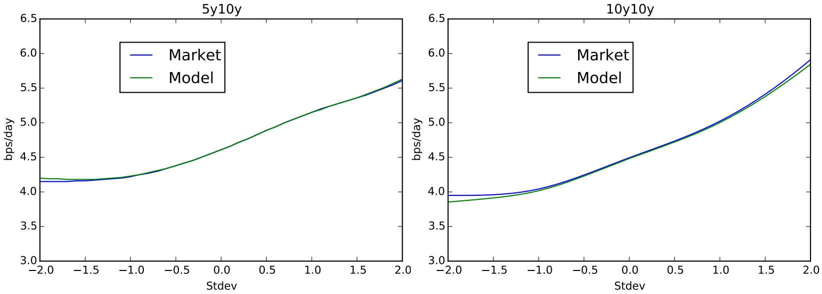

Figure: Swaption smile with maturities 5Y/10Y and tenor of 10Y compared to implied volatilities (EUR, 15-April-2016). $N = 2^{12}$ particles, 2^{15} simulations.

Extensions to multi-dimensional Cheyette

- A multi-dimensional Cheyette model:

$$dx_t = \cdot dt + \Sigma(t, x_t) \cdot dW_t$$

$\Sigma(t, x_t)$ is $N \times N$, W_t a N-dimensional Brownian motion.

- Swap rate dyanmics:

$$ds_t^{\alpha,\beta} = \left(\nabla_x s_t^{t,t+\theta}\right) \Sigma(t, x_t) \cdot dW_t^{\alpha,\beta}$$

- Take $\Sigma(t, x_t) = (\nabla_x s_t^{t,t+\theta})^{-1} \sigma(t, s_t^{t,t+\theta}) \Phi(t)$ where $\sigma(t, s_t^{t,t+\theta})$ is a scaling function and $\Phi(t)$ a deterministic $N \times N$ matrix.

- Calibration condition is:

$$\sigma(t, K)^2 \mathrm{Tr}\left(\Phi(t)^\dagger \cdot \Phi(t)\right) = \sigma_{\mathrm{loc}}(t, K)^2 + 2\frac{\Xi(t, K)}{\partial_K^2 C^{\mathrm{mkt}}(t, K)}$$

Extensions to Libor market models

- Swap rate dynamics under LLM:

$$ds_t^{\alpha,\beta} = \sum_{i=\alpha+1}^{\beta} \frac{\partial s_t^{\alpha,\beta}}{\partial L_t^i} \Sigma_t^i \cdot dW_t^{\alpha,\beta},$$

Σ_t^i is the instantaneous volatility of Libor $L_t^i \equiv L(t, T_i, T_{i+1})$.

- Take

$$\Sigma_t^i = \left(\frac{\partial s_t^{t,t+\theta}}{\partial L_t^i} \right)^{-1} \sigma(t, s_t^{t,t+\theta}) \Phi_i(t)$$

Open questions

- Existence of local vol: existence of solution of Mckean-Vlasov type SDE.
- Calibration on Vol cube: how to parametrize ?

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