Stochastic invariance of closed sets with non-Lipschitz coefficients

B. Bouchard

Ceremade - Univ. Paris-Dauphine, PSL Research University

Joint work with E. Abi Jaber (Ceremade and Axa-IM) and C. Illand (Axa-IM)

Problem

Aim of this work

$(b,\sigma):\mathbb{R}^d\to\mathbb{R}^d\times\mathbb{M}^d$ continuous, b and $\|\sigma\sigma^\top\|^{\frac{1}{2}}$ with linear growth.

$$X = x + \int_0^{\cdot} b(X_s)ds + \int_0^{\cdot} \sigma(X_s)dW_s$$

(weak solution)

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Necessary and sufficient conditions for the existence of a solution $X \in \mathcal{D}$, given $x \in \mathcal{D}$ a closed set, i.e. $\mathcal{D}$ is stochastically invariant.

Litterature

□ **General answer for "smooth coefficients" :** Friedman [6], Doss [3], Bardi and Goatin [4] and Bardi and Jensen [5] (2nd order normal cone). Da Prato and Frankowska [1] and Buckdahn et al. [7] (first order normal cone), Tappe [10] (jump diffusions).

□ **Affine or polynomial models using specific treatments :** Filipović and Mayerhofer [5], Filipović and Larsson [4] (polynomial diffusions), Cuchiero *et al.* [10] (affine processes on the cone of symmetric semi-definite matrices), Spreij and Veerman [9] (affine diffusions).

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$\Rightarrow$ We want a general answer using the first order normal cone covering smooth and non-smooth coefficients (e.g. $\sigma(x) = \sqrt{x}$)

Replace $\sigma \in C_b^{1,1}$ by $\sigma\sigma^\top \in C_{\text{loc}}^{1,1}$
and use the first order normal cone.

The regular case : $\sigma \in C^{1,1}_{loc}$

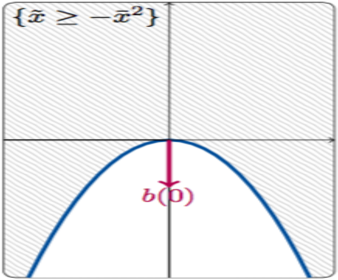
Da Prato and Frankowska [1]
and Buckdahn, Quincampoix, Rainer & Teichmann [7]

Characterization in the regular case

Thm : Assume that $\sigma \in C^{1,1}_{loc}$. $\mathcal{D}$ is stochastically invariant if and only if

$$\sigma(x)^\top u = 0 \text{ and } \langle u, b(x) - \frac{1}{2}\sum_{j=1}^{d} D\sigma^j(x)\sigma^j(x)\rangle \leq 0, \ \forall\, x \in \mathcal{D} \text{ and } u \in \mathcal{N}^1_{\mathcal{D}}(x),$$

where $\mathcal{N}^1_{\mathcal{D}}(x) := \left\{u \in \mathbb{R}^d : \langle u, y - x\rangle \leq o(\|y - x\|), \forall\, y \in \mathcal{D}\right\}$ is the first order normal cone at x.



Necessary condition in the regular case

Suffices to check that $\phi(X) \leq 0$ for $\phi : y \mapsto \langle u, y - x \rangle - \frac{\kappa}{2}\|y - x\|^2$, for some $\kappa > 0$.

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a. Apply Itô's Lemma and Girsanov theorem to get

$$0 \geq \int_0^t [\mathcal{L}\phi(X_s) + n\|D\phi(X_s)\sigma(X_s)\|^2]ds + \int_0^t D\phi(X_s)\sigma(X_s)dW_s^n.$$

Take expectation under $\mathbb{P}^n$, divide by t and $t \to 0$:

$$D\phi(x)\sigma(x) = 0 \ \Leftrightarrow \sigma(x)^\top u = 0.$$

Necessary condition in the regular case

b. Apply Itô's Lemma twice :

$$
\begin{aligned}
0 &\geq \int_0^t \mathcal{L}\phi(X_s)ds + \int_0^t D\phi(X_s)\sigma(X_s)dW_s \\
&= \int_0^t \mathcal{L}\phi(X_s)ds + \int_0^t \left[D\phi(x)\sigma(x) + \int_0^s \mathcal{L}(D\phi\sigma)(X_u)du \right] dW_s \\
&\quad + \int_0^t \int_0^s D(D\phi\sigma)(X_u)\sigma(X_u)dW_u dW_s
\end{aligned}
$$

Necessary condition.

Use Cheridito, Soner & Touzi [8], Bruder [6], Buckdahn et al. [7] ($d = 1$ here). Since X is Hölder continuous and the functions are continuous :

$$0 \geq \mathcal{L}\phi(x)t + \underbrace{\int_0^t [\mathcal{L}\phi(X_s) - \mathcal{L}\phi(x)]ds}_{o(t)} + \underbrace{\int_0^t \int_0^s \mathcal{L}(D\phi\sigma)(X_u)dudW_s}_{O(t^{\frac{3}{2}-\varepsilon})}$$
$$+ D(D\phi\sigma)(x)\sigma(x)\frac{W_t^2 - t}{2}$$
$$+ \underbrace{\int_0^t \int_0^s [D(D\phi\sigma)(X_u)\sigma(X_u) - D(D\phi\sigma)(x)\sigma(x)]dW_u dW_s}_{O(t^{1+\eta})}.$$

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& + D(D\phi\sigma)(x)\sigma(x)\frac{W_t^2 - t}{2} \\
& + \underbrace{\int_0^t \int_0^s [D(D\phi\sigma)(X_u)\sigma(X_u) - D(D\phi\sigma)(x)\sigma(x)]dW_u dW_s}_{O(t^{1+\eta})}.
\end{aligned}
$$

Divide by t and use that $\liminf_{t\to 0} W_t^2/t = 0$ ($d = 1$ here) :

$$
0 \geq \mathcal{L}\phi(x) - \frac{1}{2}D(D\phi\sigma)(x)\sigma(x) = \langle u, b(x) - \frac{1}{2}D\sigma(x)\sigma(x)\rangle.
$$

Exemple of irregular case : $\sigma(x) = \sqrt{|x|}$

The case $\sigma(x) = \sqrt{|x|}$ and $X \geq 0$

We consider

$$X = 0 + \int_0^{\cdot} a(b - X_s)ds + \int_0^{\cdot} \sqrt{|X_s|}dW_s.$$

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Can not apply Itô's Lemma to $\sqrt{|X|}$...
but can just take expectation to get

$$0 \leq \lim_{t \to 0} \frac{1}{t}\mathbb{E}[\int_0^t a(b - X_s)ds] = ab,$$

that turns out to be necessary and sufficient....

Regular vs Irregular case

□ Regular case : want to keep the contribution of the diffusion part $\Rightarrow$ “pathwise analysis”.

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☐ Regular case : want to keep the contribution of the diffusion part $\Rightarrow$ "pathwise analysis".

☐ Irregular case : want to kill the contribution of the diffusion part $\Rightarrow$ "analysis in expectation".

Need to find a way to
kill the "irregular" directions and keep the "regular" ones.

The general case

$\sigma\sigma^\top$ can be extended into a $C^{1,1}_{\mathrm{loc}}$ function C

A toy example

We consider

$$X = 0 + \int_0^{\cdot} b(X_s)ds + \int_0^{\cdot} \underbrace{\begin{pmatrix} \sigma^1(X_s) & 0 \\ 0 & \sqrt{X_s^2} \end{pmatrix}}_{\sigma(X_s)} dW_s.$$

with $\mathcal{D} = \mathcal{D}^1 \times \mathbb{R}_+$.

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with $\mathcal{D} = \mathcal{D}^1 \times \mathbb{R}_+$.

Take $x^2 = 0$. We want to kill the irregular part :

$$\begin{aligned} 0 \geq & \mathbb{E}_{\mathcal{F}_T^{W^1}} \left[\int_0^t \mathcal{L}\phi(X_s)ds + \int_0^t D\phi(X_s)\sigma(X_s)dW_s \right] \\ = & \int_0^t \mathbb{E}_{\mathcal{F}_T^{W^1}}[\mathcal{L}\phi(X_s)]ds + \int_0^t \mathbb{E}_{\mathcal{F}_T^{W^1}}[D_1\phi(X_s)\sigma^1(X_s)]dW_s^1 \end{aligned}$$

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and obtain (by the same arguments as before)

$$0 \geq \mathcal{L}\phi(x) - \frac{1}{2}D_1(D_1\phi\sigma^1)(x)\sigma^1(x) = \langle u, b(x) - \frac{1}{2}\sigma^1(x)D_1\sigma^1(x)\rangle.$$

The case of a regular spectral decomposition

Assume that $\sigma\sigma^\top$ can be extended into a $C^{1,1}_{\mathrm{loc}}$ function :

$$C = Q\mathrm{diag}\left[\lambda_1, \ldots, \lambda_r, 0, \ldots, 0\right] Q^\top$$

with $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_r(x) > 0$ and $Q(x)Q(x)^\top = I_d$, $r \leq d$.

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Then, $\bar\sigma : y \mapsto \bar Q(y)\bar\Lambda(y)^{\frac{1}{2}}$ is $C^{1,1}(N(x))$, in which $\bar Q := [q_1 \cdots q_r \; 0 \cdots 0]$ and $\bar\Lambda = \mathrm{diag}[\lambda_1, ..., \lambda_r, 0, ..., 0]$.

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Moreover,

$$\begin{aligned} 0 &\geq \int_0^t \mathcal{L}\phi(X_s)ds + \int_0^t D\phi(X_s)(Q\Lambda^{\frac{1}{2}}Q^\top)(X_s)dW_s \\ &= \int_0^t \mathcal{L}\phi(X_s)ds + \int_0^t D\phi(X_s)(Q\Lambda^{\frac{1}{2}})(X_s)d\bar W_s \end{aligned}$$

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and take expectation given $\sigma((\bar{W}^1_s, \ldots, \bar{W}^r_s), s \leq T)$ to get as above

$$0 \geq \langle u, b(x) - \frac{1}{2}\sum_{j=1}^{d} DC^j(x)(CC^+)^j(x)\rangle.$$

The general case

Take $A_\varepsilon = Q(x)\mathrm{diag}[1-\varepsilon, (1-\varepsilon)^2, \ldots, (1-\varepsilon)^d]Q(x)^\top$ so that

$C_\varepsilon(x) = Q(x)\mathrm{diag}[(1-\varepsilon)\lambda_1(x), (1-\varepsilon)^2\lambda_2(x), \ldots, (1-\varepsilon)^d\lambda_d(x)]Q(x)^\top$

has distinct non-zero eigenvalues and one can apply the above to

$$X_\varepsilon := A_\varepsilon X = A_\varepsilon x + \int_0^\cdot b_\varepsilon(X_s^\varepsilon)ds + \int_0^\cdot C_\varepsilon(X_s^\varepsilon)^{\frac{1}{2}} dW_s$$

with respect to $\mathcal{D}_\varepsilon := A_\varepsilon \mathcal{D}$. Then, $\varepsilon \to 0$.

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with respect to $\mathcal{D}_\varepsilon := A_\varepsilon \mathcal{D}$. Then, $\varepsilon \to 0$.

Thm : Assume that $\sigma\sigma^\top = C$ on $\mathcal{D}$ for some $C \in C^{1,1}_{\mathrm{loc}}$. Then, $\mathcal{D}$ is stochastically invariant if and only if

$$\begin{cases} C(x)u = 0 \\ \langle u, b(x) - \frac{1}{2}\sum_{j=1}^d DC^j(x)(CC^+)^j(x)\rangle \leq 0 \end{cases}$$

for every $x \in \mathcal{D}$ and for all $u \in \mathcal{N}^1_{\mathcal{D}}(x)$.

Extension to jump diffusions by E. Abi Jaber [1]

For the diffusion with jumps

$$X = x + \int_0^{\cdot} b(X_s)ds + \int_0^{\cdot} \sigma(X_s)dW_s + \int_0^{\cdot} \int \rho(X_{s-}, z)\left(\mu(ds, dz) - F(dz)ds\right),$$

the conditions become

$$\begin{cases} x + \rho(x,z) \in \mathcal{D}, \text{ for } F\text{-almost all } z, \\ \int |\langle u, \rho(x,z)\rangle| F(dz) < \infty, \\ \sigma(x)^\top u = 0, \\ \langle u, b(x) - \int \rho(x,z)F(dz) - \frac{1}{2}\sum_{j=1}^d DC^j(x)(CC^+)^j(x)\rangle \le 0, \end{cases}$$

for all $x \in \mathcal{D}$ and $u \in \mathcal{N}_{\mathcal{D}}(x)$

An example
Polynomial diffusions on parabolic concave state space

Polynomial diffusions

Definition : X is a polynomial diffusion on $\mathcal{D}$ if :

(i) There exist $\bar{b}^i, \widetilde{b}^i \in \mathbb{R}$, $0 \le i \le 2$, and $A^i \in \mathbb{S}^2$, $1 \le i \le 5$, such that $b : x \mapsto b(x) := (\bar{b}(x), \widetilde{b}(x)) \in \mathbb{R}^2$ and $C : x \mapsto C(x) \in \mathbb{S}^2$ have the following form :

$$\begin{cases} \bar{b}(x) & = \bar{b}^0 + \bar{b}^1 \bar{x} + \bar{b}^2 \widetilde{x}, \\ \widetilde{b}(x) & = \widetilde{b}^0 + \widetilde{b}^1 \bar{x} + \widetilde{b}^2 \widetilde{x}, \\ C(x) & = A^0 + A^1 \bar{x} + A^2 \widetilde{x} + A^3 \bar{x}^2 + A^4 \bar{x}\widetilde{x} + A^5 \widetilde{x}^2, \end{cases}$$

for all $x = (\bar{x}, \widetilde{x}) \in \mathcal{D}$.

(ii) $C(x) \in \mathbb{S}^d_+$, for all $x \in \mathcal{D}$.

When $A^i = 0$ for all $3 \le i \le 5$, we say that X is an affine diffusion.

Parabolic concave state space

We consider :

$$\mathcal{D} = \{(\bar{x}, \widetilde{x}) \in \mathbb{R}^2, \widetilde{x} \geq -\bar{x}^2\}.$$

Our conditions are equivalent to

$$\begin{cases} C(x) = C_{11}(x) \begin{pmatrix} 1 & -2\bar{x} \\ -2\bar{x} & 4\bar{x}^2 \end{pmatrix}, \\ \langle u, b(x) \rangle - \frac{\mathbf{1}_{\{C_{11}(x) \neq 0\}}}{2(4\bar{x}^2+1)} \left[2\bar{x}\partial_u (C_{11} - C_{22})(x) + (1 - 4\bar{x}^2)\partial_u C_{12}(x) \right] \geq 0, \end{cases}$$

for all $\bar{x} \in \mathbb{R}$, $x = (\bar{x}, -\bar{x}^2)$ and $u = (2\bar{x}, 1)^\top \in -\mathcal{N}^1_{\mathcal{D}}(x)$.

Necessary and sufficient conditions

- No affine solution unless it has no diffusion part or leaves on the boundary !

Necessary and sufficient conditions

□ No affine solution unless it has no diffusion part or leaves on the boundary !

□ For polynomial diffusions, $\mathcal{D}$ is invariant if and only if there exist $\alpha, \beta \geq 0$ such that either one the following conditions holds :

(a)

$$C(x) = \begin{pmatrix} \alpha & -2\alpha\bar{x} \\ -2\alpha\bar{x} & (4\alpha+\beta)\bar{x}^2 + \beta\widetilde{x} \end{pmatrix}, \text{ for all } x = (\bar{x}, \widetilde{x}) \in \mathcal{D},$$

(b) $\bar{b}^2 = 0$ and

$$\begin{cases} \widetilde{b}^2 < 2\bar{b}^1 \quad \text{and} \quad (\widetilde{b}^1 + 2\bar{b}^0)^2 \leq 4(-\widetilde{b}^2 + 2\bar{b}^1)(\widetilde{b}^0 + \alpha) \\ \text{or} \\ \widetilde{b}^2 = 2\bar{b}^1, \quad \widetilde{b}^1 = -2\bar{b}^0 \quad \text{and} \quad \widetilde{b}^0 \geq -\alpha. \end{cases}.$$

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